AD-A079 881 AIR FORCE INST OF TECH WRIGHT-PATTERSON AFB OH SCHOO--ETC F/G 22/3
DETERMINATION OF THE MINIMUM DELTA V TRANSFER TRAJECTORY FROM A--ETC(U)
DEC 79 T E WEIMER

UNCLASSIFIED AFIT/GA-AA/79D-11 NL

1 OF 2
AD
A079881

AFIT/GA/AA/79D-11

DETERMINATION OF THE MINIMUM ΔV TRANSFER TRAJECTORY FROM A LOW EARTH ORBIT TO A STABLE ORBIT AROUND THE LAGRANGIAN POINT L4 IN A RESTRICTED FOUR-BODY SYSTEM

THESIS

AFIT/GA/AA/79D-11

THERON E. WEIMER
CAPTAIN USAF

Approved for public release; distribution unlimited

14

AFIT/GA/AA/79D-11

Delta

6

DETERMINATION OF THE MINIMUM [redacted]V TRANSFER
TRAJECTORY FROM A LOW EARTH ORBIT TO A STABLE
ORBIT AROUND THE LAGRANGIAN POINT L4 IN A
RESTRICTED FOUR-BODY SYSTEM.

9 Master's THESIS,

PRESENTED TO THE FACULTY OF THE SCHOOL OF ENGINEERING
OF THE AIR FORCE INSTITUTE OF TECHNOLOGY
AIR UNIVERSITY (ATC)
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

12 114

A

BY

10 THERON E. WEIMER
CAPTAIN USAF
GRADUATE ASTRONAUTICS

11 DEC[redacted]79

Approved for public release; distribution unlimited

012225

Preface

I would like to express my sincere appreciation to several people for their assistance in the preparation of this thesis. First of all, I would like to deeply thank my thesis advisor, Professor William E. Wiesel, for his guidance, advice and encouragement. He has been an inspiration to me from the beginning. Without his expert and professional help this work would not have been possible. I would also like to thank Dr. Robert A. Calico and Professor James Rader for their time and interest in reviewing my report. I would also like to extend a special thanks to my typist, Dorothy Smith, whose proficiency and skill were a tremendous asset in rendering this study to its final form. Finally, I would like to express a very deep thanks to my wife, Diane, and son, Tad, for their understanding and inspiration throughout this project.

Contents

List of Figures

List of Tables

List of Symbols

A	Matrix whose components are the partial derivatives of the equations of motion with respect to the states
a	semi-major axis
α	angle between the x_I axis and x_e axis
$\dot{\alpha}$	rotation rate of B_{em} about B_{sem}
B	correction matrix relating $\Delta \underline{x}$ to launch parameter adjustments
B_{em}	Earth-Moon Barycenter
B_{sem}	Sun-Earth-Moon Barycenter
C	Matrix relating changes in launch conditions and changes of the initial state
$\delta x(t_0)$	change in initial state vector
ΔV	total velocity change, $\Delta V_1 + \Delta V_2$
ΔV_1	impulsive velocity change to leave Earth orbit
ΔV_2	impulsive velocity change to enter Wheeler's orbit
$\Delta \underline{x}$	error vector at t_1
$\underline{e}_1, \underline{e}_2$	unit vectors in the x_e, y_e coordinate frame
E_t	total energy for Hohmann transfer
G	gravitational constant
$\underline{i}, \underline{j}$	unit vectors in the x_E, y_E coordinate frame
I	identity matrix
L	the lagrangian
L4,L5	stable, equilibrium libration points of the restricted three-body problem

M	mass of an orbiting body
M_1	mass of Earth
M_2	mass of Moon
M_c	mass of satellite
M_s	mass of Sun
μ	M_2, mass of Moon
$\mu_\oplus$	gravitational parameter for Earth
ODE	ordinary differential equation integration package
ω	angular rotation rate of satellite as defined by Kepler's Third Law
ϕ	launch angle
Φ	state transition matrix
q	generalized coordinates
$\underline{r}_1$	radius vector from Earth to satellite
$\underline{r}_2$	radius vector from Moon to satellite
$\underline{r}_3$	radius vector from Sun to satellite
$\underline{r}$	radius vector from B_{em} to satellite
$\underline{R}_B$	radius vector from B_{sem} to B_{em}
$\underline{R}_I$	radius vector from B_{sem} to satellite
$\underline{R}_s$	radius vector from B_{sem} to Sun
$\underline{R}_{SE}$	$\lvert R_B \rvert + \lvert R_s \rvert$
T	kinetic energy
T_E	one sidereal year

T_p	$2\pi/\omega$, period of revolution of a satellite
T_s	synodic period of the Moon
t	time
t_0	time of initial conditions of the states
TOF	time of flight
θ	angle between x_E axis and x_e axis
$\dot{\theta}$	rotation rate of Moon about the Earth
V	potential energy
$\underline{V}$	velocity of satellite with respect to x_e, y_e coordinate frame
V_c	circular velocity of satellite around the Earth
$\underline{V}_E$	velocity of satellite with respect to x_E, y_E coordinate frame
$\underline{V}_I$	velocity of satellite with respect to x_I, y_I coordinate frame
x, y	location of satellite with respect to x_e, y_e coordinate frame
$\underline{X}$	state vector with components x_1, x_2, x_3, x_4
x_e, y_e	rotating coordinate frame axes where the x_e axis is along the line joining the Earth and Moon
X_E, Y_E	rotating coordinate frame about the B_{sem}
X_I, Y_I	inertial coordinate frame relative to the stars and centered at B_{sem}
$\underline{X}(0)$	initial state vector with components $x_0, y_0, \dot{x}_0, \dot{y}_0$
$\underline{X}(t_0)$	$\underline{X}(0)$
$\underline{X}(t)$	state vector at any time, t
$\underline{X}_o(t)$	initial reference trajectory
$\tilde{x}(t_1)$	x,y coordinates of target points

Abstract

In this report, the equations of motion for a satellite in restricted four-body motion are derived and examined. An algorithm is developed that produces a minimum ΔV transfer trajectory from a low Earth orbit to establishment in Wheeler's stable periodic orbit around L4. Both a Hohmann transfer and an infinite velocity, straight line transfer are examined as initial conditions for integration of the equations of motion. Transfer trajectories are determined to ten points around Wheeler's orbit so that a minimum ΔV trajectory can be determined. Two types of minimum ΔV values were found. One type is a true minimum value, the other type is a limiting value beyond which the target orbit can not be reached. Results are presented on a transfer trajectory that gives a minimum ΔV from the points examined. From these results a more accurate transfer location is estimated.

DETERMINATION OF THE MINIMUM ΔV TRANSFER TRAJECTORY FROM A LOW EARTH ORBIT TO A STABLE ORBIT AROUND THE LAGRANGIAN POINT L4 IN A RESTRICTED FOUR-BODY SYSTEM

I Introduction

Background

In the past few years the idea of developing space colonies has been gaining strength. NASA (Ref 6) has recently looked at the feasibility of establishing space settlements. They determined that space colonization is desirable because of the hope it offers humanity. It has been suggested that colonies can provide a solution to many earthly problems ranging from energy sources to overcrowding population growth. The location of such colonies has been a major subarea of study. The majority of work in this area has been to find periodic orbits in the restricted three-body problem, with the Earth, Moon and space colony forming the three-body model. Recently, however, there have been attempts to investigate the motion of the satellite in a restricted four-body model by adding the gravitational forces of the Sun. The finding of periodic orbits in the four-body model causes questions concerning transfers to these orbits. The need for transporting personnel, materials and equipment to the site at an efficient and economic

rate must be investigated. That is the main purpose of this paper.

From a military viewpoint, the orbits that are suitable for nearby colonies may also be likely locations for military outposts or large military structures. The portion of the Earth visible at any one time from the orbit and its proximity to both the Earth and the Moon are factors that would be important to the military strategist. For these reasons, orbits in the vicinity of the Lagrangian points L4 and L5 are of military importance.

Dr. Gerard K. O'Neill (Ref 8) of Princeton University is this country's leading authority in the field of space colonies and their suitable locations. In his book he states that L5 has lost its status as the probably host site for the first space colony. However, very recently, J. E. Wheeler (Ref 10) investigated the existance of periodic, stable orbits near L4 and L5. His finding of a stable 1/1 resonant orbit at L4 revitalizes the concept of a colony or military outpost near that point. This 1/1 orbit means that the satellite revolves around L4 once for every time the Moon revolves around the Earth.

T. A. Heppenheimer (Ref 5) and his colleagues B. O'Leary and D. Kaplan (Ref 7) have also examined the problem of colony location and transfer trajectories. Heppenheimer found a 2/1 resonant orbit around the Earth

that could be reached from L2 by a Hohmann transfer sequence. He did not, however, examine orbits or transfer trajectories in the neighborhood of L4 or L5 because current mass-driver capabilities would not allow a catcher near either of these points. In their work together, O'Leary, Kaplan and Heppenheimer found a transfer trajectory from L2 that circles the Moon twice and then passes through L1 to enter a 2:1 resonant orbit around the Earth. Again, L4 and L5 were not examined. As the accuracy of mass-driven launchers improves, orbits for manufacturing facilities near L4 or L5 become more feasible. As this happens, the need for examining transfer trajectories to this area will increase.

As the need for examining transfer trajectories grows, techniques for finding these orbits must be developed. Much of today's work is done with Hohmann transfers, such as Heppenheimer's work (Ref 5). Hohmann transfers provide the minimum ΔV required for a transfer between two coplanar, circular orbits in a two-body system. In a restricted four-body system, such as exists here, Hohmann transfers may not produce the minimum ΔV transfer. The necessity for developing a better method definitely exists.

Problem and Scope

The problem is to determine the minimum velocity change required to transfer from a low Earth orbit to the orbit around L5 found by Wheeler. The very restricted four-body model was used. In this model, the Earth and Moon travel in circles around their barycenter. This barycenter and the Sun follow circular orbits around the Earth-Moon-Sun barycenter. The motion of the colony, as it moves through this system, is governed by the gravitational forces created by each of the other three bodies. All motion in the system is limited to one plane.

The Lagrangian Points, L4 and L5, near which Wheeler found his orbit, are equilibrium points in the restricted three-body problem. At these points the sum of the gravitational forces of the Earth and Moon, plus any centrifugal force on the satellite equals zero. If a satellite were placed at one of these points with the proper initial velocity it will remain fixed relative to the Earth and Moon. When the gravitational force of the Sun is added to the model, these points are no longer equilibrium points. As shown in Fig 1, the points L4 and L5 form a equilateral triangle with the Earth. Therefore, any transfer trajectory to one will be the mirror image of the other. The only time that this is not true is when the perturbing effects of other heavenly bodies are introduced. These perturbing

effects are not considered here, hence L4 and L5 can be used interchangeably.

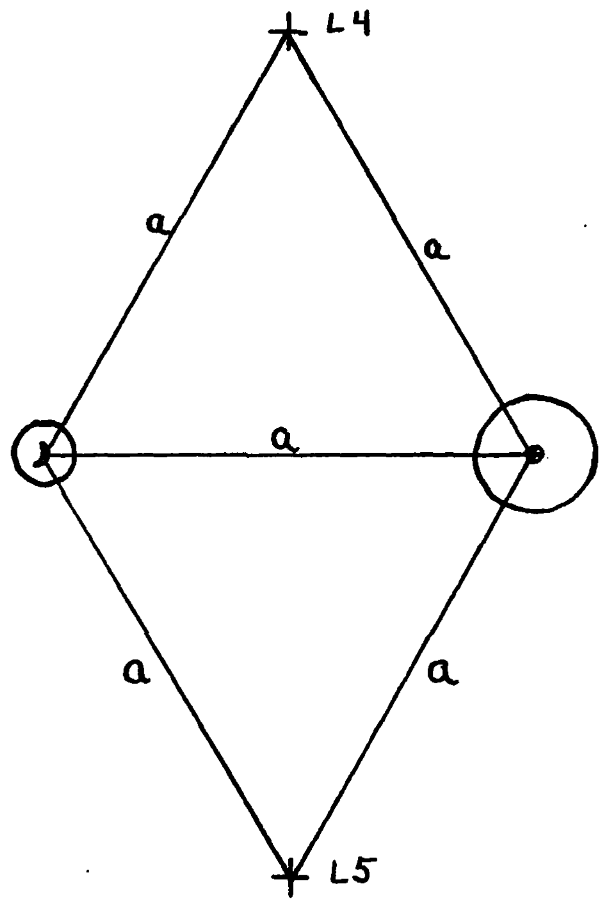


Fig 1. Location of L4 & L5 with respect to Earth (⊕) and Moon (☽)

II Problem Analysis

The primary background work that this thesis is based upon was done by J. E. Wheeler (Ref 10) as an AFIT thesis. Wheeler's dynamical model and basic problem assumptions were used as the primary foundations for this paper. From his thesis a 1/1 orbit was selected as a target for a transfer trajectory from a low Earth orbit.

Assumptions

Following are the basic assumptions used in the very restricted four-body model.

1) The Sun, Earth, and Moon are considered to be point masses.
2) The mass of the satellite is negligible when compared to the other three bodies and, therefore, it does not affect their motion.
3) The gravitational effects of the other planets in our solar system can be ignored.
4) The motion of all four bodies is limited to one plane.
5) Earth and Moon move in circular orbits about their barycenter at a constant rotation rate.

6) The Earth-Moon barycenter moves in a circular orbit about the Earth-Moon-Sun barycenter at a constant rate.

7) The satellite moves in a circular orbit about the Earth at a constant rate.

The assumed circular orbits are actually slightly elliptical. The Moon's actual eccentricity is 0.0549 (Ref 2:325), and the Earth's eccentricity is 0.0167 (Ref 2:360). The slight errors that result from assuming circular orbits will be on the order of the eccentricities of the orbits. The assumption of planer motion is also not exact. The Moon's orbit is inclined 5°8' (Ref 2:325) from the Earth's orbital plane. This will also cause small errors to occur. These errors indicate that Wheeler's stable, periodic 1/1 orbit isn't exactly periodic and minute perturbations will exist. These small perturbations will also be present in any transfer trajectory to Wheeler's orbit found from the circular restricted four-body model. Based on the previously mentioned assumptions, however, any motion found from the circular restricted four body model should be nearly identical to the actual motion.

Astronautical Constants

Before continuing with the development of the equations of motion, the parameters used in the analysis must be defined. In Fig 2, if the mass of the Moon, M_2, is set to some arbitrary value, μ, and the sum of M_1 (Earth) + M_2 is equal to 1, then

$$M_1 = 1 - \mu \tag{1}$$

if the moment arms are now summed, it can be shown that the distance from the Earth and the Moon to the barycenter are μ and $1 - \mu$, respectively. The solution for μ is then

$$\mu = \frac{M_2}{M_1 + M_2} \tag{2}$$

The value of μ used is

$$\mu = .0121396054$$

(Ref 9:517)

The synodic period of the Moon, T_s, will appear throughout this report. It corresponds to $\dot{\theta}$ which is specified as

$$\dot{\theta} = \frac{2\pi}{T_s} = 1.0 \tag{3}$$

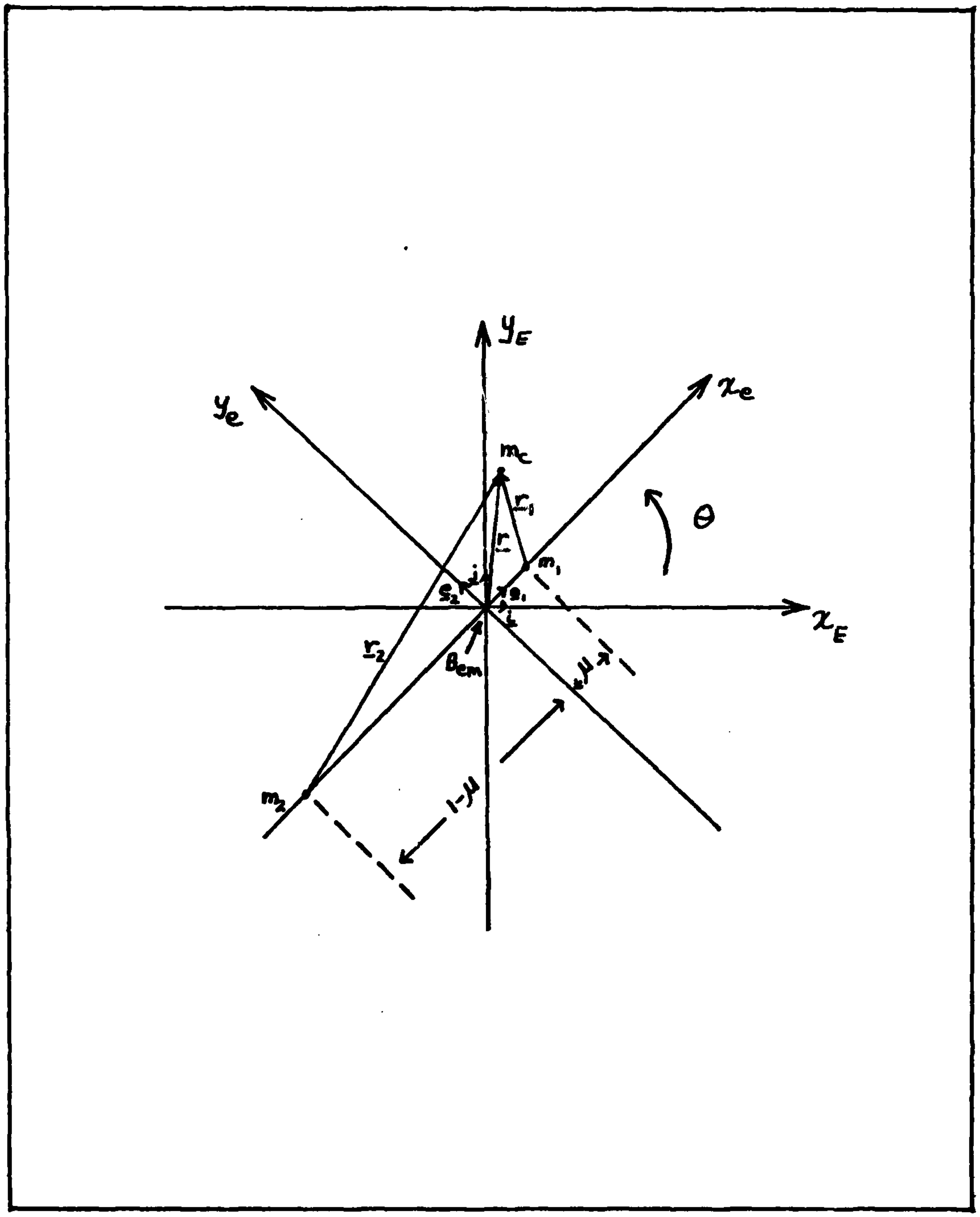


Fig 2. Earth-Moon System

with

$$T_s = 29.5305882 \text{ days}$$

Now referring to Fig 3,

$$\frac{M_1 + M_2}{M_s} = \frac{1}{328900.12}$$

(Ref 4:144)

(4)

or

$$M_s = 328900.12$$

T_E, the sidereal year, will be used to determine $\dot{\alpha}$,

$$\dot{\alpha} = \frac{2\pi}{T_E} \tag{5}$$

with

$$T_E = 365.25636556 \text{ days}$$

(Ref 3:334)

since

$\dot{\theta} = 1$ in normalized units, then $\dot{\alpha}$ becomes

$$\dot{\alpha} = \left(\frac{2\pi}{365.25636556}\right) \cdot \left(\frac{29.5305882}{2\pi}\right)$$

or

$$\dot{\alpha} = .0808489351$$

Again, summing moment arms in Fig 3. gives

$$R_S \; M_S = R_B \; (M_1 + M_2) \tag{6}$$

or

$$\frac{R_B}{R_S} = \frac{M_S}{M_1 + M_2} = 328900.12 \tag{7}$$

Applying Kepler's Third Law

$$T_p = \frac{2 \; \pi}{\omega} = 2 \; \pi \; (a^3 \; / \; GM)^{\frac{1}{2}} \tag{8}$$

or

$$\omega^2 \; a^3 = GM \tag{9}$$

The rotation rate of the Earth-Moon system with respect to an inertial frame is

$$\omega = \dot{\theta} + \dot{\alpha} \tag{10}$$

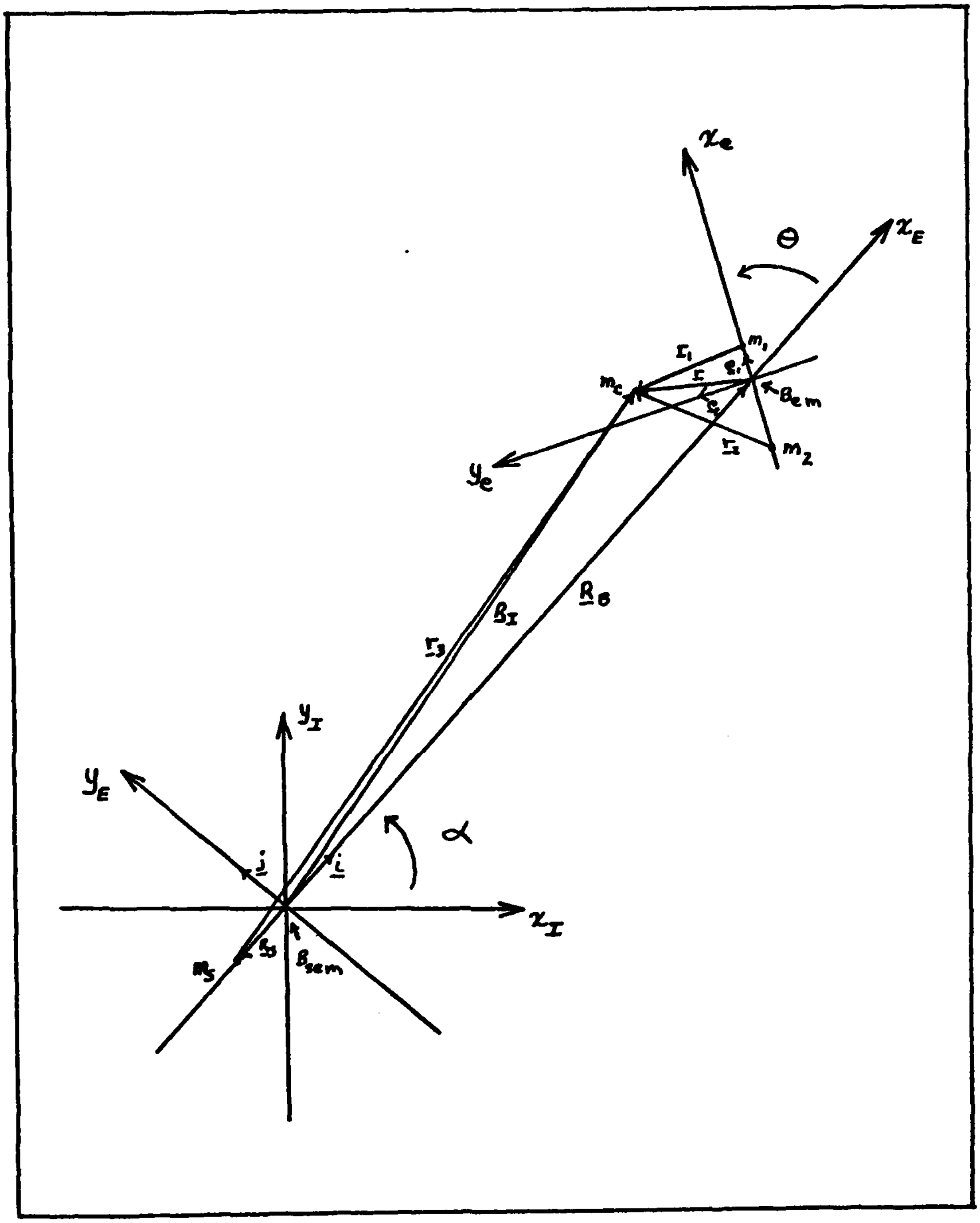


Fig 3. Four-Body Configuration

So, looking at just the Earth-Moon system

$$(\dot{\theta} + \dot{\alpha})^2 a^3 = G (M_1 + M_2) = G (1)$$

With a equal to the distance from the Earth to the Moon, 1 Du, then

$$(1 + \dot{\alpha})^2 = G = 1.16823442 \quad (11)$$

Another relationship for R_B and R_S can now be solved for using Kepler's Law again. Concentrating $(M_1 + M_2)$ at B_{EM} and rotating about B_{SEM} at an ω equal to $\dot{\alpha}$ gives

$$(\dot{\alpha}^2) \quad (R_B + R_S)^3 = G (M_S + M_1 + M_2) \quad (12)$$

or

$$(R_B + R_S) = R_{SE} = 388.8202849 \quad (13)$$

and from Eq. (7),

$$\frac{R_B}{R_S} = 328900.12$$

Combining Eq.s. (7) and (13) gives

$$R_B = 388.8191027 \qquad R_S = .0011821799 \tag{14}$$

Equations of Motion

All motion will be referred to the inertial reference frame x_I, y_I in $\underline{e}_1, \underline{e}_2$ unit vectors. From Fig 3,

$$\underline{r} = x\,\underline{e}_1 + y\,\underline{e}_2 \tag{15}$$

$$\dot{\underline{r}} = \underline{V} = \dot{x}\,\underline{e}_1 + \dot{y}\,\underline{e}_2 \tag{16}$$

$$\underline{V}_E = \underline{V} + (\dot{\theta} \times \underline{r}) = (\dot{x} - \dot{\theta} y)\underline{e}_1 + (\dot{y} + \dot{\theta} x)\underline{e}_2 \tag{17}$$

$$\begin{aligned}\underline{V}_I = \underline{V}_E + (\dot{\alpha} \times \underline{R}_I) = {} & [\dot{x} - (\dot{\theta} + \dot{\alpha})y + \dot{\alpha}\, R_B \sin\theta]\underline{e}_1 \\ & + [\dot{y} + (\dot{\theta} + \dot{\alpha})x + \dot{\alpha}\, R_B \cos\theta]\underline{e}_2\end{aligned} \tag{18}$$

The kinetic energy is

$$T = \tfrac{1}{2} M_C\, (V_I \cdot V_I)$$

or

$$T = \tfrac{1}{2} M_C \{[\dot{x} - (\dot{\theta} + \dot{\alpha})y + \dot{\alpha} R_B \sin\theta]^2$$

$$+ [\dot{y} + (\dot{\theta} + \dot{\alpha})x + \dot{\alpha} R_B \cos\theta]^2\} \tag{19}$$

The potential energy is

$$V = -\sum_{i=1}^{3} \frac{GM(M_C)}{r_i} \tag{20}$$

or

$$V = -GM_C \{\frac{1-\mu}{[(x-\mu)^2 + y)^2]^{\frac{1}{2}}} + \frac{\mu}{[(1-\mu+x)^2 + y^2]^{\frac{1}{2}}}$$

$$+ \frac{M_S}{[(x+R_{SE}\cos\theta^2 + (y-R_{SE}\sin\theta)^2]^{\frac{1}{2}}}\}$$

The Lagrangian is $L = T - V$. To solve for the equations of motion, Lagranges's equations are employed:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0,\ i = 1,2 \tag{21}$$

and the generalized coordinates are $q_1 = x$, $q_2 = y$.

$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$ yields the x equation of motion

$$\ddot{x} = 2\dot{y}(\dot{\theta}+\dot{\alpha}) + (\dot{\theta}+\dot{\alpha})^2 x + R_B\, \dot{\alpha}^2 \cos\theta - \frac{G(1-\mu)(x-\mu)}{[(x-\mu)^2+y^2]^{3/2}}$$

$$- \frac{G\mu(1-\mu+x)}{[(1-\mu+x)^2+y^2]^{3/2}} - \frac{GM_s\,(R_{SE}\cos\theta + x)}{[(R_{SE}\cos\theta+x)^2+(y-R_{SE}\sin\theta)^2]^{3/2}}$$

(22)

Similarly, $\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0$ yields the y equation of motion

$$\ddot{y} = -2\,\dot{x}\,(\dot{\theta}+\dot{\alpha})+(\dot{\theta}+\dot{\alpha})^2 y - \dot{\alpha}R_B \sin\theta - \frac{G(1-\mu)y}{[(x-\mu)^2+y^2]^{3/2}}$$

$$- \frac{G\,\mu\,y}{[(1-\mu+x)^2+y^2]^{3/2}}$$

$$- \frac{M_s G\,(y - R_{SE}\sin\theta)}{[(R_{SE}\cos\theta+x)^2+(y-R_{SE}\sin\theta)^2]^{3/2}}$$

(23)

To put these equations into first-order form a state space approach was used. The state vector $\underline{X}$ was defined as

$$\underline{X} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{Bmatrix}$$

(24)

The following first-order differential equations result:

$$\dot{X}_1 = \dot{x} \tag{25}$$

$$\dot{X}_2 = \dot{y} \tag{26}$$

$$\dot{X}_3 = \ddot{x} \tag{27}$$

$$\dot{X}_4 = \ddot{y} \tag{28}$$

Note that from Eq. (3), $d\theta = dt$ or $\theta = t$, the time, in Eqs. (22) and (23). Therefore, these equations vary with time and may be represented by the equation:

$$\dot{\underline{x}}\ (t) = \underline{f}\ [\underline{x}\ (t),\ t] \tag{29}$$

This equation says that the first order non-linear equations are functions of the states themselves and of time.

For small displacements from the states at any time, t, Eq. (29) becomes:

$$\dot{\underline{x}}\ (t) + \delta\dot{\underline{x}}\ (t) = \underline{f}\ [(\underline{x}(t) + \delta x\ (t)\ ),\ t] \tag{30}$$

By the use of a Taylor series expansion and Eq. (29), Eq. (30) becomes:

$$\delta\dot{\underline{x}}\ (t) = \frac{\partial \underline{f}}{\partial \underline{x}(t)}\ \delta\underline{x}\ (t) \tag{31}$$

This equation relates small displacements in the states to the equations of motion through a square matrix A(t), where

$$A(t) = [\frac{\partial \underline{f}}{\partial \underline{x}}] = [\frac{\partial \dot{\underline{x}}_i}{\partial \underline{x}_j}] \tag{32}$$

In this case A is a 4 x 4 matrix that is used to linearize small displacements. Its complete listing can be found in Appendix A.

<u>Equations of Variation</u>

The method used by Wheeler to produce periodic motion is almost the same as the method used to produce a transfer trajectory to a specific point. In order to accomplish this, the initial conditions must be related to the state of the system at the time, t. Starting with an initial trajectory $\underline{x}_0$ (t), the states can be perturbed to give a general solution close to $\underline{x}_0$ (t).

$$\underline{x}(t) \simeq \underline{x}_0(t) + \delta \underline{x}(t) \tag{33}$$

Expanding in a Taylor series gives

$$\underline{x}(t) \simeq \underline{x}_0(t) + \frac{\partial \underline{x}(t)}{\partial \underline{x}(t_0)}\bigg|_{x_0} \delta x(t_0) + \text{higher order terms}$$

Utilizing the state transition matrix, $\Phi(t, t_0)$ which is defined by

$$\delta \underline{x}(t_0) = \Phi\ (t,\ t_0)\ \delta \underline{x}(t_0) \tag{34}$$

or

$$\Phi(t,t_0) = \frac{\partial \underline{x}(t)}{\partial \underline{x}(t_0)} \tag{35}$$

This matrix relates changes in the states at t to changes in the initial states at t_0. Differentiating Eq. (34) yields.

$$\delta \dot{\underline{x}}(t) = \dot{\Phi}(t,t_0)\delta x_0(t_0) + \Phi(t,t_0)\ \delta \dot{x}(t_0)$$

or

$$\delta \dot{\underline{x}}(t) = \dot{\Phi}(t,t_0)\delta x\ (t_0) \tag{36}$$

since $\delta \dot{\underline{x}}(t_0)$ equals zero. Placing Eqs. (34) and (36) into Eq. (31) gives

$$\dot{\Phi}(t,t_0) = A(t)\ \Phi(t,t_0) \tag{37}$$

The last equation allows the propogation of Φ along the trajectery to be determined.

Simultaneous numerical integration of the equations of motion and variation will provide the needed information on a particular solution at the final time, so that a transfer trajectory can be produced.

III Transfer Trajectory

The problem of finding an optimum transfer trajectory to Wheeler's orbit can be divided into three sub-problems. The first part is a two-point boundary value problem attempting to arrive at a designated point on Wheeler's orbit at a specified time. The second part of the problem consists of minimizing the delta V for the transfer trajectory found in the two-point boundary value problem. The final part of the problem is to repeat parts one and two at representative points around Wheeler's orbit. From the collection of transfer trajectories the minimum cost transfer to Wheeler's orbit can be determined.

Two-Point Boundary Value Problem

This two-point boundary value problem is bounded by a known starting point $r(t_0)$ and a known ending point $r(t_1)$. The orbit velocity, $v(t_1)$, is also known. The velocity at launch, $v(t_0)$, needs to be found such that a transfer is made from $r(t_0)$ to $r(t_1)$. There are three launch parameters that can be specified to help determine the initial conditions for launch. See Fig 4. The first parameter is the impulsive velocity change, ΔV_1, that is added to the satellites orbital velocity around the Earth to cause it to leave its low Earth orbit. The second parameter is the launch angle, ϕ, that determines where

in the orbit the impulse should be added. The final parameter is the launch time, t_0. This time tells where the Sun is in relation to the other three bodies. Wheeler's orbit is phase dependent, so the position of the influencing bodies is very important. The satellite must arrive at a point on Wheeler's orbit when the Earth, Moon and Sun are in the required positions for that point on the orbit. That makes the time of arrival, t_0, very important, but for a specified ΔV and ϕ a specific t_0 is required to arrive at the target at t_1.

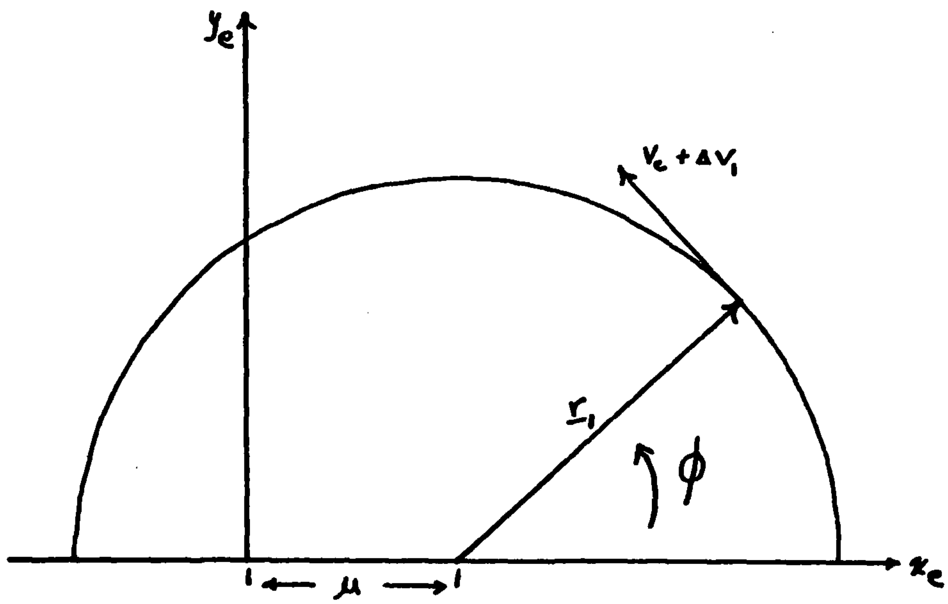


Fig 4. Launch parameters

The method used to produce a transfer trajectory that solves the two-point boundary value problem is very similar to that used by Wheeler to find his periodic orbit. The end point is not the same as the starting point, as in Wheeler's problem, but the concept is the same. By starting with an estimate of the initial state, $\underline{x}(t_0)$, the states can be propogated along the trajectory by Eq. (29).

$$\dot{\underline{x}}(t) = f\,[\underline{x}(t),\, t]$$

At $x(t_1)$, it is found that the trajectory does not close on the desired point, $\tilde{x}(t_1)$, but is some $\Delta\underline{x}$ distance away from the target.

The error vector is then defined by

$$\Delta\underline{x} = \tilde{x}(t_1) - x(t_1) \tag{38}$$

To close the trajectory on $\tilde{x}(t_1)$, $\Delta\underline{x}$ must be driven to zero. This can be done by adding a $\delta x(t_0)$ which will generate a $\delta x(t_1)$ as expressed by Eq. (34)

$$\delta x(t_1) = \Phi(t_1,t_0)\ \delta x(t_0)$$

So, to obtain closure,

$$\Delta x = \delta x(t_1) \tag{39}$$

or

$$\Delta x = \Phi(t_1,t_0)\ \delta x(t_0) \tag{40}$$

However, the initial state, $x(t_0)$, is dependent upon the launch parameters, L. Therefore,

$$\delta x(t_0) = C\delta L \tag{41}$$

Where C is a 2×4 matrix relating changes in the launch parameters and changes in the initial state.

So,

$$\Delta x = \Phi C \delta L \tag{42}$$

or

$$\delta L = B^{-1}\Delta x \tag{43}$$

where

$$B = \Phi C. \tag{44}$$

From this last equation, corrections to the launch parameters can be solved for in terms of known quantities.

An algorithm for solving this two-point boundary value

problem and producing a transfer trajectory is summarized as follows:

Equations of motion: $\dot{\underline{x}}(t_0) = \underline{f}\ [x(t\ ),t]$

where $\underline{x}(t_0)$ equals initial conditions.

Equations of variation: $\dot{\Phi} = A\Phi, \Phi(t_0,t_0) = I$

At $t_0 = 0$, a guess is made of the launch parameters, L, which determine the initial conditions, $x(t_0)$. Integrating the equations of motion and the equations of variation to some time, $t = t_1$, the resulting quantities are $\underline{x}(t_1)$ and $\Phi(t_1,t_0)$. Utilizing Eq. (44), B can be found. Also, knowing that $\Delta x = \tilde{x}(t_1) - x(t_1)$, δL can be computed from Eq. (43). The next estimate of the launch parameters is then

$$L_1 = L - \delta L \tag{45}$$

which will give us $\underline{x}_1(t_0)$. This algorithm will then repeat itself until $\Delta\underline{x}$ is smaller than some convergence criteria, thus defining a transfer trajectory. Appendix B has a complete development of the B matrix that was used. Appendix C is a computer listing of this algorithm.

Minimum ΔV

The second part of the problem is to minimize the total velocity change required to establish the satellite in

Wheeler's orbit at a particular target point. The object of this sub-problem is displayed in Fig 5.

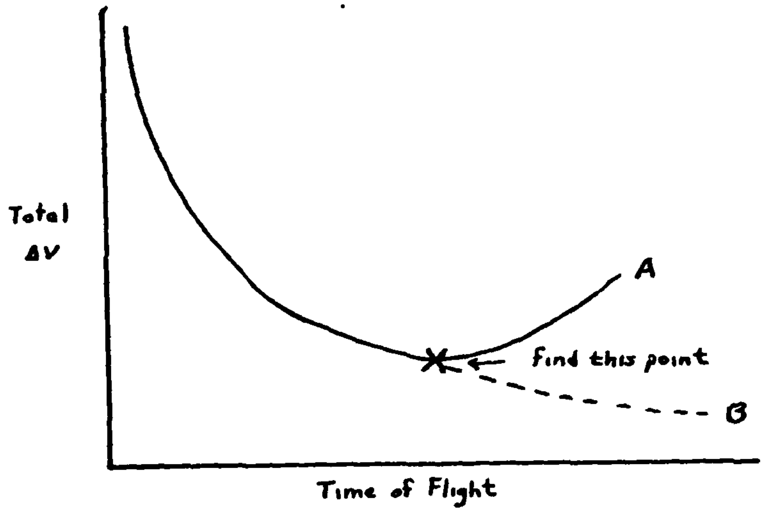


Fig 5. Minimize total ΔV

This figure displays the type of results that are expected. As the TOF approaches zero on the graph, the required velocity to get to the point would have to increase. The limiting case would be for an infinite velocity, which would result from an infinite ΔV. An infinite velocity would produce a straight-line transfer and it would happen instantaneously. So, as TOF approaches zero, ΔV would approach infinity.

The minimum ΔV point on the graph may be characterized by two types of behavior. It may truly be a minimum

value and past the corresponding TOF ΔV would increase. (Curve A). However, there is another possibility. The minimum ΔV may be a limiting value. Any TOF beyond this point would produce a lower ΔV, but the resulting velocity would not be sufficient to reach the target. (Curve B). A Hohmann transfer is an example of this in two body motion. It provides the minimum ΔV trajectory that will complete the transfer (Fig 6). Anything less falls short of its target (Fig 7).

To produce a curve similar to Fig 5, small changes in the TOF used in the algorithm that solves the two-point boundary value problem will be made to produce changes in the total ΔV. This process is repeated until the minimum is bracketed or it becomes apparent that a limiting value exists.

If $TOF(1) < TOF(2) < TOF(3)$, $\Delta V(1) > \Delta V(2)$ and $\Delta V(2) < \Delta V(3)$, then a minimum exists between TOF(1) and TOF(3). From this point a bisection routine is used to find the minimum ΔV. The bisection algorithm is as follows:

1) $TOF = (TOF(1) + TOF(3))/2$

2) Use algorithm for boundary value problem to find ΔV for TOF.

3) This TOF replaces TOF(1) or TOF(3), whichever has the largest associated ΔV. e.g. If $\Delta V(1) < \Delta V(3)$, then $TOF \rightarrow TOF(3)$ and the algorithm is repeated.

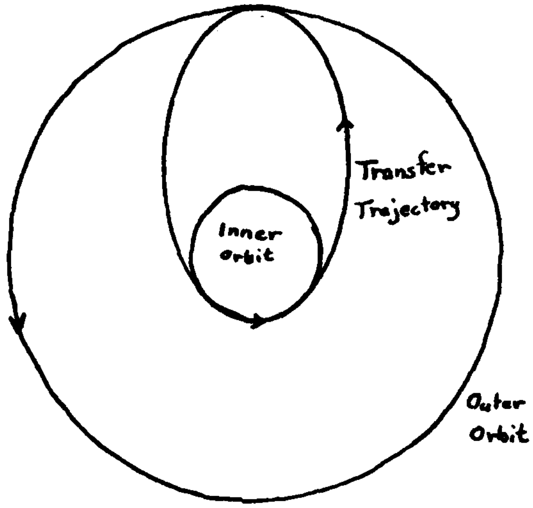


Fig 6. Hohmann Transfer
(minimum ΔV)

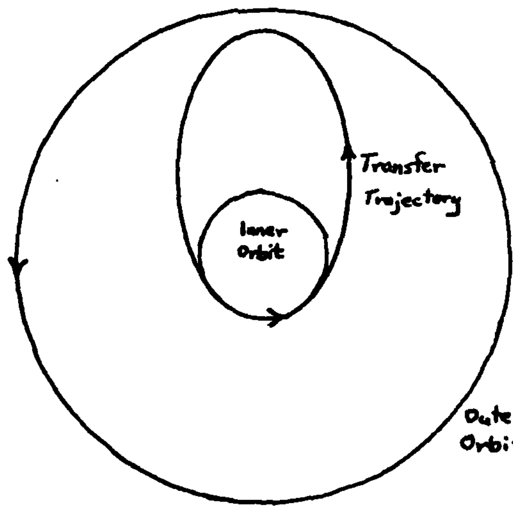


Fig 7. ΔV less than Hohmann transfer

4) This procedure will continue until the difference between the existing minimum ΔV and the ΔV for TOF is less than a specified convergence criteria.

If a limiting value appears to exist, the original procedure of increasing TOF by small amounts is continued until the target point can no longer be reached to the desired accuracy.

<u>Minimum transfer</u>

After finding the minimum transfer trajectory to one point on Wheeler's orbit, the process is repeated for a nearby point. This method continues until the entire orbit has been examined. It must be remembered that Wheeler's orbit is phase dependent, and therefore so are the transfer trajectories. That is why the entire orbit must be examined. By utilizing only a representative number of points around the orbit a curve relating ΔV to time of arrival (position) on the orbit can be developed. See Fig 8. For this curve, minimum points can be examined further for more precise results. Because Wheeler's orbit is periodic, the points at each end of the curve will be the same.

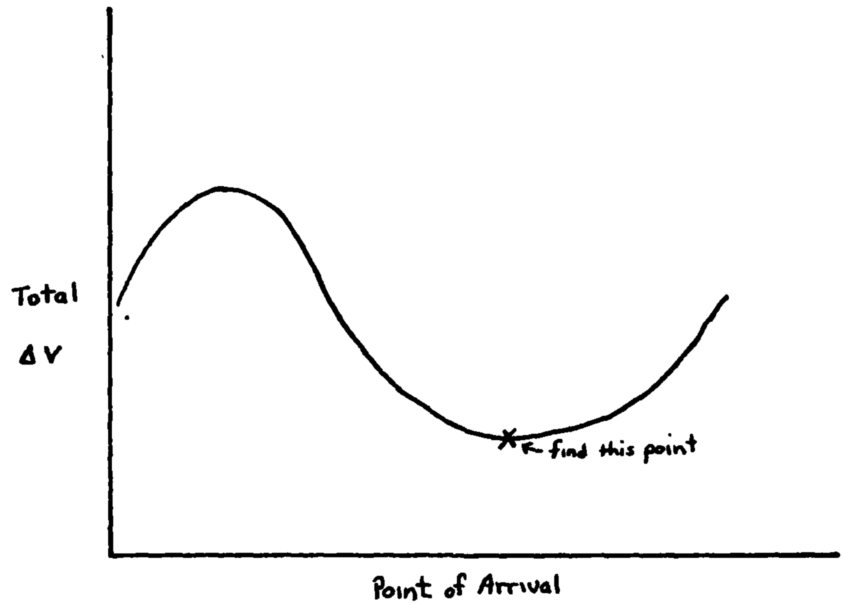


Fig 8. Minimum ΔV

Initial Conditions

To begin integration for this problem, approximate initial conditions must be determined. Launch conditions must be established that will allow the satellite to arrive at the target point when the Earth, Moon and Sun are in the proper position to allow maneuvering into the orbit. This can be done by specifying the values of Θ and α. See Fig 3. However, $\Theta = t$ (Ref 10:19) and, by the same reasoning, specifying time will also determine α. It is, therefore, necessary that the values of Θ and α at $t = 0$ be known to correctly relate the four bodies. Initial

conditions on the satellite must also be determined. Its orbital altitude and velocity are necessities for formulation of the proper launch conditions. The final initial conditions are the launch parameters themselves. The launch angle, ϕ, and impulsive velocity change, ΔV_1, are required for integration of the transfer trajectory.

Because Wheeler's orbit is phase dependent, the positions of the Earth, Moon and Sun determine when a point is on the orbit and when it is not. Therefore, it is imperative that the satellite arrive at the target point when the three governing bodies, (Earth, Moon and Sun), are in positions that allow that point to be part of Wheeler's periodic stable orbit. The time of arrival, t_1, is related to the launch time, t_0, by the time of flight, TOF.

$$TOF = t_1 - t_0 \tag{46}$$

To ensure compatability of time measurements, Wheeler's initial conditions of $\Theta = \alpha = 0$ at $t = 0$ were used. They are diagrammed in Fig 9. From this base, t_1, which expresses the Earth, Moon, Sun positioning, can be determined for any point on the target orbit. Placing a specified TOF and t_1 into Eq. (46) yields a launch time, t_0. This value can then be used in the equations of motion (Eqs. 25-28), for the integration. The target orbit and

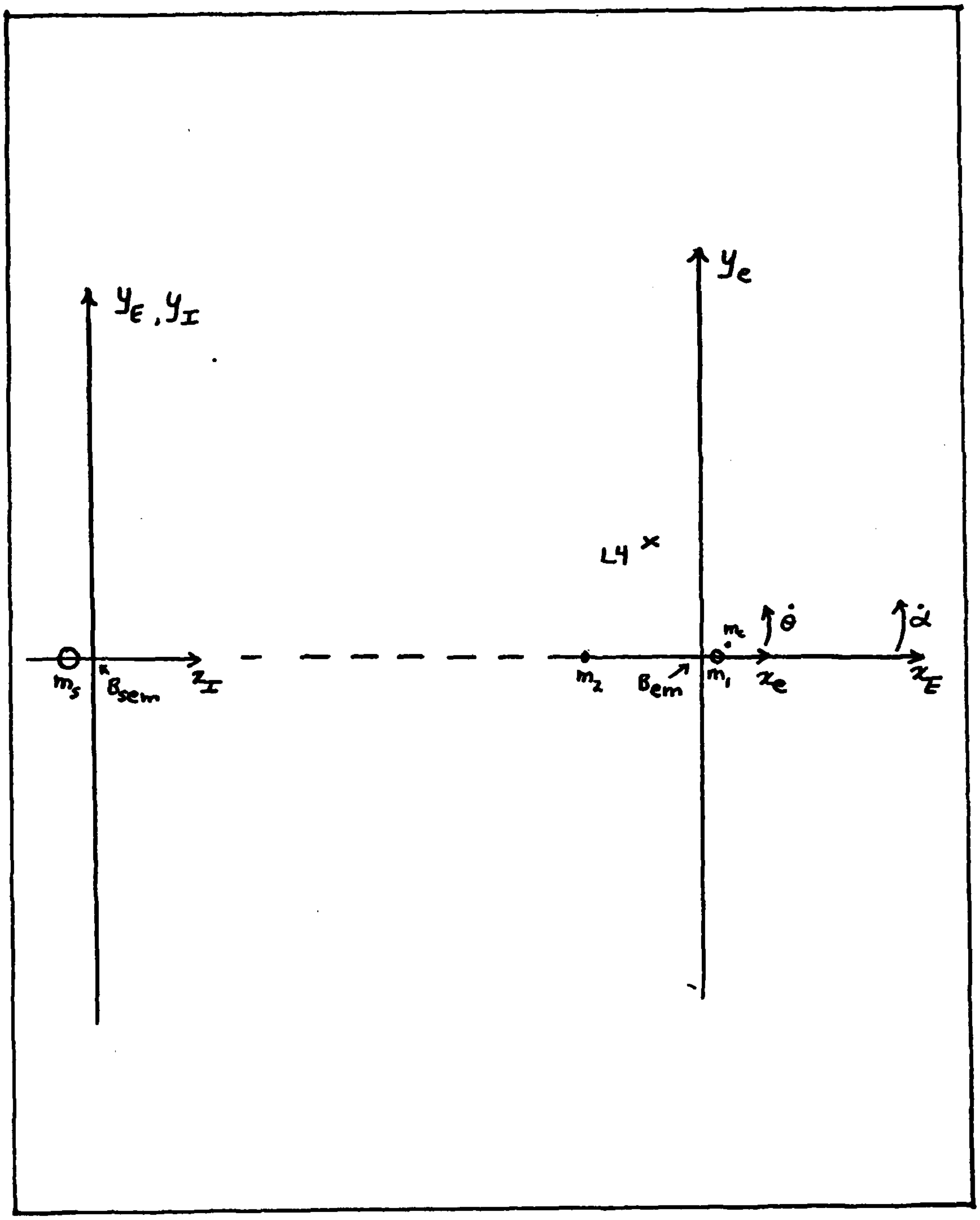


Fig 9. Initial Condition Configuration of Four-Body Model

t_1's for selected points are shown in Fig 10 and Table 1. These were generated from an adaptation of Wheeler's program which is listed in Appendix C.

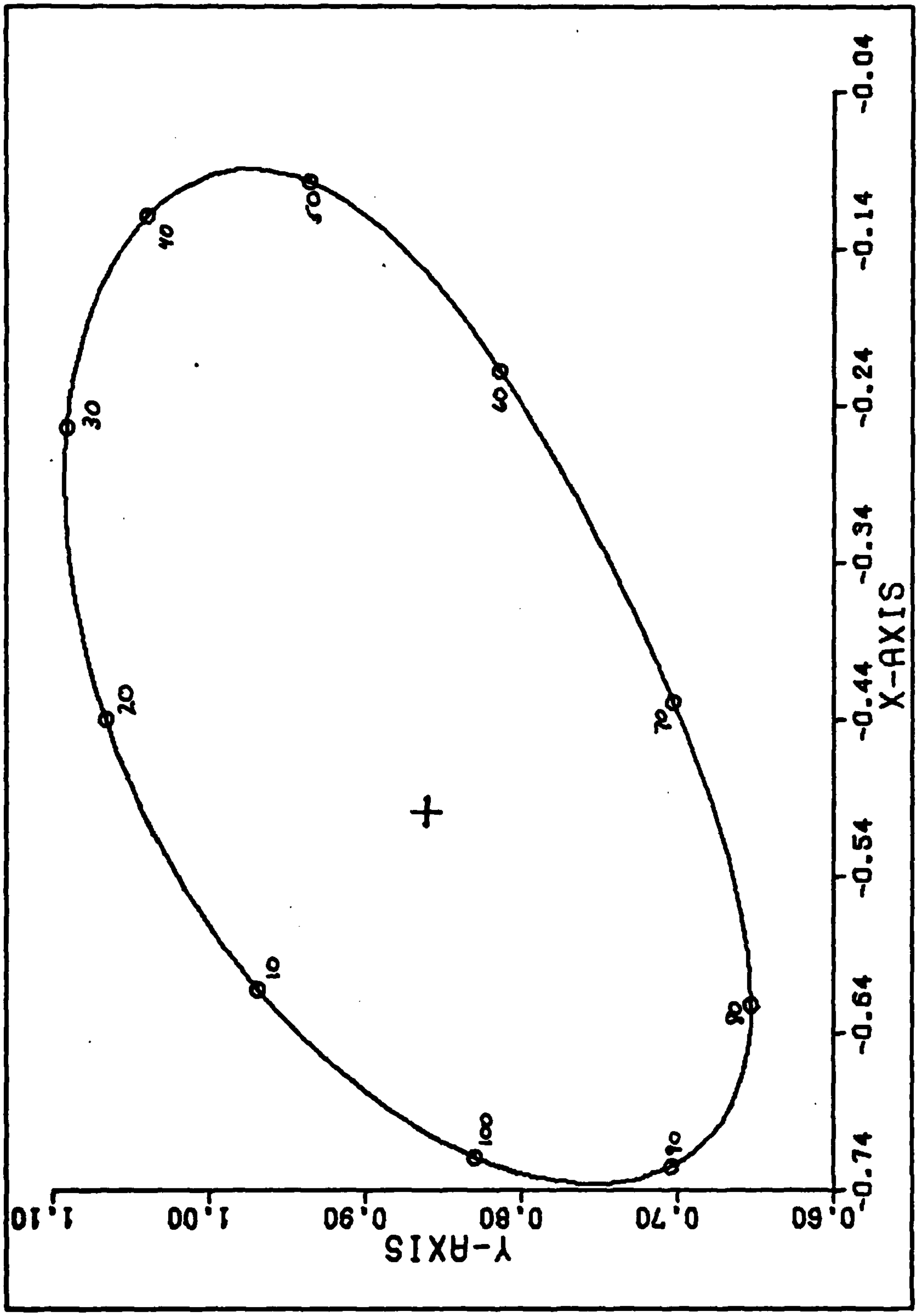
Four conditions must be established for the satellite before its launch position can be determined. Looking at Eqs. (B-1, 2, 7, 8), it is apparent that the orbital altitude $-r_1$, orbital velocity $-V_c$, launch angle $-\phi$, and velocity change $-\Delta V$, are necessary to compute the initial conditions of the satellite for the transfer integration. The orbital altitude selected was $r_1 = .0170110431$ Du in Wheeler's units where 1 Du is the Earth-Moon distance of 384,400 Km (1:441). This corresponds to a 100 mile orbit which is a commonly used low Earth orbit.

The orbital velocity of the satellite is dependent upon the altitude chosen. Assuming two-body motion because of the satellites proximity to the Earth, the orbital velocity is given by

$$V_c = (\mu_\oplus / r_1)^{\frac{1}{2}} \qquad \text{(Ref 2:165)} \qquad (47)$$

where the Earth's gravitational parameter is defined as

$$\mu_\oplus = GM_1 \qquad \text{(Ref 2:14)} \qquad (48)$$



Target Orbit Around L5
Fig 10

Table 1

Target Points On Orbit Around L4

Pt	Time	X	XDOT	Y	YDOT
10	.62831853070	-.6265820293608	.2225823196025	.9558641295779	.2065634249742
20	1.25663706140	-.4587250848339	.2980404139519	1.059612446891	.1144292755284
30	1.88495559210	-.2716104412929	.2804640199443	1.092545703858	-.01145432442989
40	2.51327412280	-.1286410538959	.1570727714585	1.047943452510	-.1244391508112
50	3.14159265350	-.09321334187446	-.05486875527559	.9473721315225	-.1854511712815
60	3.76991118420	-.1998009153287	-.2740889061399	.8259280751101	-.1946820792853
70	4.39822971490	-.4073707541026	-.3531797486920	.7121732359906	-.1545103295533
80	5.02654824560	-.6062887269687	-.2599216846324	.6535319881410	-.01747024308636
90	5.65486677630	-.7197139834979	-.09557856578724	.6950575481862	.1426301345351
100	6.28318530700	-.7241878255950	-.07948061605492	.8156863939862	.2243800772464

with $G = 1.16823442$

(11)

and $M_1 = 1 - \mu = .985818095$

(1)

then, $\mu_{\oplus} = 1.15166663$

So, placing these values in Eq. (47)

$$V_c = 8.22806889 \quad Du/Tu$$

Even if a four body model is used, the close Earth region is dominated by the Earth, and the optimum transfer orbit should not depend strongly on V_c.

With V_c and r_1, determined, it is now necessary to come up with a good initial guess for ϕ, ΔV_1 and TOF. Two possible methods may be used. One method is to ignore the effects of the Moon and the Sun, making the transfer a two body problem. Under these conditions a Hohmann transfer could be used to provide the initial ϕ, ΔV_1, and also the TOF. A minimum ΔV_1 Hohmann transfer in a non-rotating frame would require that the satellite be launched from low Earth orbit when it preceded the target by 180°. See Fig 10. Using point 80 from Fig 10 and Table 1,

$$x = -.6062887269693$$

$$y = .6535319881417$$

From Fig 11,

$$\tan \alpha = y/(x + \mu) \tag{48}$$

$$\alpha = 46.58°$$

$$\phi = 360 - \alpha = 313.42° \tag{49}$$

or

$$\phi = 5.470196073 \text{ radians}$$

However, this is a rotating reference frame. Once the TOF is determined a correction can be added to ϕ for the amount of rotation during the travel of the satellite.

TOF is determined by utilizing

$$TOF = \pi \left(a_t^3 / \mu\right)^{1/2} \qquad \text{(Ref 2:165)} \tag{50}$$

$$a_t = (r_1 + r_2)/2 = .458382633 \qquad \text{(Ref 2:164)} \tag{51}$$

$$TOF = .908507911 \text{ Tu}$$

This is approximately 4.26993184 days, as 1 Tu equals 4.69993908 days (Ref 10:75). This is based upon the fact that the period of the stable orbit is the same as the synodic period of the Moon at 2π Tu's.

Using the TOF, a correction to ϕ may now be calculated. During .908507911 Tu's the system will rotate .908507911 radians. This value should be added to the ϕ for no rotation to give a corrected value,

$$\phi = 5.470196073 + .908507911$$

$$\phi = .095518677 \text{ radians}$$

Because the total energy of the Hohmann transfer is constant,

$$E_t = -\mu_\oplus / (r_1 + r_2) \qquad \text{(Ref 2:164)} \tag{52}$$

$$r_1 = 1.70110431 \times 10^{-2} \text{ Du}$$

$$r_2 = ((x + \mu)^2 + y^2)^{1/2} = .899754223 \text{ Du}$$

$$\mu_\oplus = 1.15166663$$

Then,

$$E_t = -1.256228473$$

The velocity at the launch point of the transfer trajectory can now be found.

$$V_1 = [2(\mu_\oplus/r + E_t)]^{1/2} \qquad \text{(Ref 2:164)} \tag{53}$$

$V_1 = 11.52778289$ Du/Tu

Then, $\Delta V_1 = V_1 - V_c = 3.299713999$ Du/Tu

The other technique used to find approximate initial conditions was to assume a straight-line transfer orbit. See Fig 12. This type of trajectory means that ϕ would be determined by where the straight-line transfer was tangent to the low Earth orbit. From Fig 12 it can be seen that

$$\sin \beta = r_1/r_2$$
$$\beta = 1.083317004^\circ \tag{54}$$

$$\tan \alpha = (y/x + \mu)$$
$$\alpha = 46.58085192 \tag{55}$$

$$\phi = 90 - \alpha + \beta$$
$$\phi = 44.5024^\circ = .776714541 \text{ rad} \tag{56}$$

To actually achieve a straight-line transfer trajectory on infinite velocity would be necessary. To represent a infinite velocity, $V_t = 1 \times 10^5$ Du/Tu's was chosen. Therefore, $\Delta V_1 \simeq 1 \times 10^5$ Du/Tu. To find TOF,

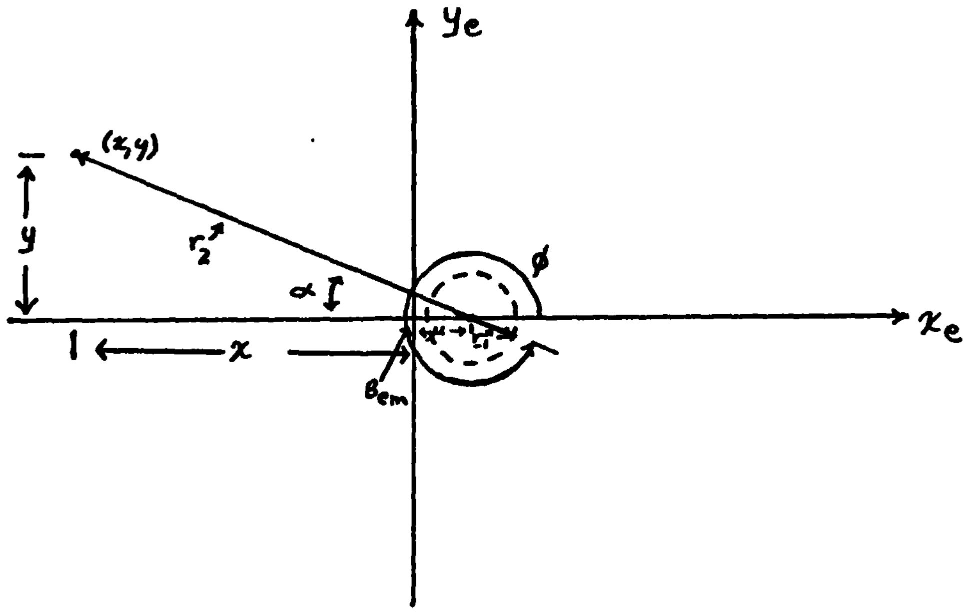


Fig 11. Hohmann Transfer

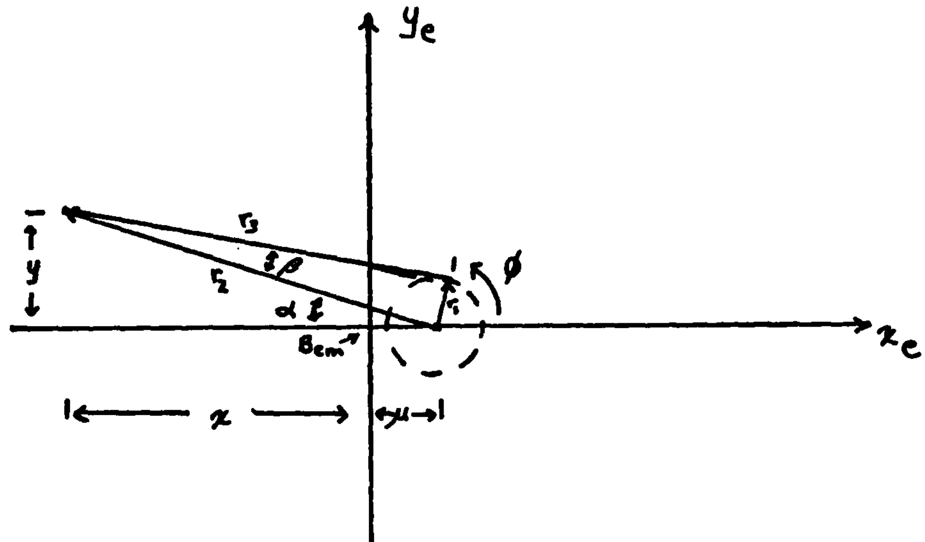


Fig 12. Tangential Trajectory

$$
\begin{aligned}
TOF &= r_3/V_t \\
&= (r_2^{\,2} - r_1^{\,2})^{1/2}/V_t \qquad (57)\\
&= 8.99690466 \times 10^{-6}\ Tu
\end{aligned}
$$

Methodolgy

The initial approach to the problem began by using a Hohmann transfer to approximate the initial transfer trajectory. The first integration was run with an accuracy of 10^{-1} Du. From these results, another integration was run with an accuracy of 10^{-2} Du. However, this accuracy could not be achieved. Plotting the transfer trajectory showed that it never approached target point 80 but appeared to be drawn away by the Sun's gravitational attraction. This would indicate that the Sun's perturbations are too large to ignore, even for determining an initial guess at a transfer trajectory.

A Hohmann transfer is not a member of the solution set to the four-body equations of motion because of the assumption that the Sun's effect can be ignored. However, a straight-line trajectory is an actual solution, even if it can not be practically achieved. Therefore, a straight-line trajectory was assumed and the algorithm was run for target point 80.

The first run for a straight-line transfer was run to 10^{-6} Du accuracy which is less than a quarter mile. This integration verified that the algorithm was at least functioning properly as it produced a transfer trajectory. Having found a transfer to point 80, it was necessary to step down ΔV until a minimum could be located. ΔV was stepped down from 10^5 to 10^4, then to 10^3 with acceptable results. For each ΔV change a new integration time was computed. Throughout these integrations the initial ϕ value used remained the same. At $\Delta V = 10^3$ the accuracy was increased to 10^{-7} and the procedure of using the results of one integration as input for the next was begun. It was also at this point that the integration time or TOF became the primary control variable. By extrapolation from previous results, a new set of ΔV_1 and ϕ were formulated for a specified TOF. This procedure was continued until the minimum ΔV was bracketed. This was indicated by an increase in ΔV from an increase in TOF. From this point on, a bisection method was utilized.

It was during the use of extrapolation procedures that accuracy became a problem. When using the extrapolated values as initial integration conditions it was sometimes necessary to drop the desired accuracy to 10^{-1} Du just to get some kind of a transfer orbit for results. Using the resulting ϕ and ΔV_1 from this orbit the accuracy could

be increased by one or two orders of magnitude until 10^{-5} was achieved. At 10^{-5} Du accuracy (about 2.5 miles) a new, larger TOF would be chosen and a new ϕ and ΔV_1 values computed.

The bisection routine used consisted of choosing a TOF midway between the two bracketing values. When the resulting ΔV was formulated it replaced the highest of the two bracketing ΔV's. This process was then repeated until the difference between the bracketing ΔV's was less than 10^{-9} Du/Tu.

During the process of finding transfer trajectories to points 10, 20, 30 and 100 it became apparent that the ΔV vs TOF curve did not have a minimum, but actually appeared to approach a limiting value as was discussed previously. The primary indication was the inability to maintain a nonsingular B matrix past a certain TOF for the desired level of accuracy. What this means is that past a certain TOF the ΔV's that are found are not large enough to propel the satellite to within the required accuracy distance of the target point. Since a bisection routine can not be used here, a new technique was required. When it became obvious that a minimum was not present and a limiting value was present, the last TOF that produced an accuracy within 10^{-9} Du was noted. To this TOF decreasing values of ΔTOF were added until 10^{-9} accuracy could be attained.

This procedure was continued until a ΔTOF of 10^{-9} produced a ΔV change of less than 10^{-9}. As an example:

1. Last TOF with 10^{-9} accuracy - 1.6 Tu
2. Add .01 Tu to get next attempted value - 1.61 Tu
3. Extrapolate from previous findings to get ϕ and ΔV_1
4. Attempt to integrate to 10^{-9} accuracy
5. If successful, repeat 1 → 4
6. If not successful, change ΔTOF to .001 - 1.601 Tu
7. Repeat 3 and 4
8. If successful repeat at that level until 10^{-9} can not be reached. (i.e. - 1.607 good, 1.608 not good)
9. Then proceed to 1.6071
10. Continue process until ΔTOF is 10^{-9} and the change in $\Delta V < 10^{-9}$

After the minimum ΔV to each arrival point has been found, a graph can be constructed that will allow an estimate of the arrival point with the minimum ΔV. (See Fig 8) If the apparent minimum is not at one of the examined points, several points near the estimated minimum may be examined to arrive at the transfer trajectory with a minimum ΔV. It must be remembered that between any two points on the target orbit there can be found another point, so this procedure could continue forever. However, the target orbit was described by 100 points and 10 points were examined. That will be the limiting spacing for this thesis.

IV Results and Discussion

Results

A transfer trajectory was found to each of the ten points examined on Wheeler's orbit. Each of these points represents an arrival time and Table 1 shows these relationships. All ten transfer trajectories can be characterized by a unique set of launch parameters and maneuver characteristics. These are tabulated in Table 2 for easy reference. Of the ten orbits, the transfer to point 10 has the lowest total ΔV.

$$\Delta V = 3.972123159 \text{ Du/Tu} = 3760.10622 \text{ m/sec}$$

Its launch parameters are

$$\phi = .724843243 \text{ radians or } 41.5204586°$$

$$\Delta V_1 = 3.305668879 \text{ Du/Tu or } 3129.22475 \text{ m/sec}$$

The time from launch to arrival at Wheeler's orbit is

$$\text{TOF} = 1.80201 \text{ Tu or } 8.46933723 \text{ days}$$

To enter Wheeler's orbit after arriving at point 10 a second burn is required. It will produce ΔV_2, which is

$$\Delta V_2 = .666454280 \text{ Du/Tu} = 630.881 \text{ m/sec}$$

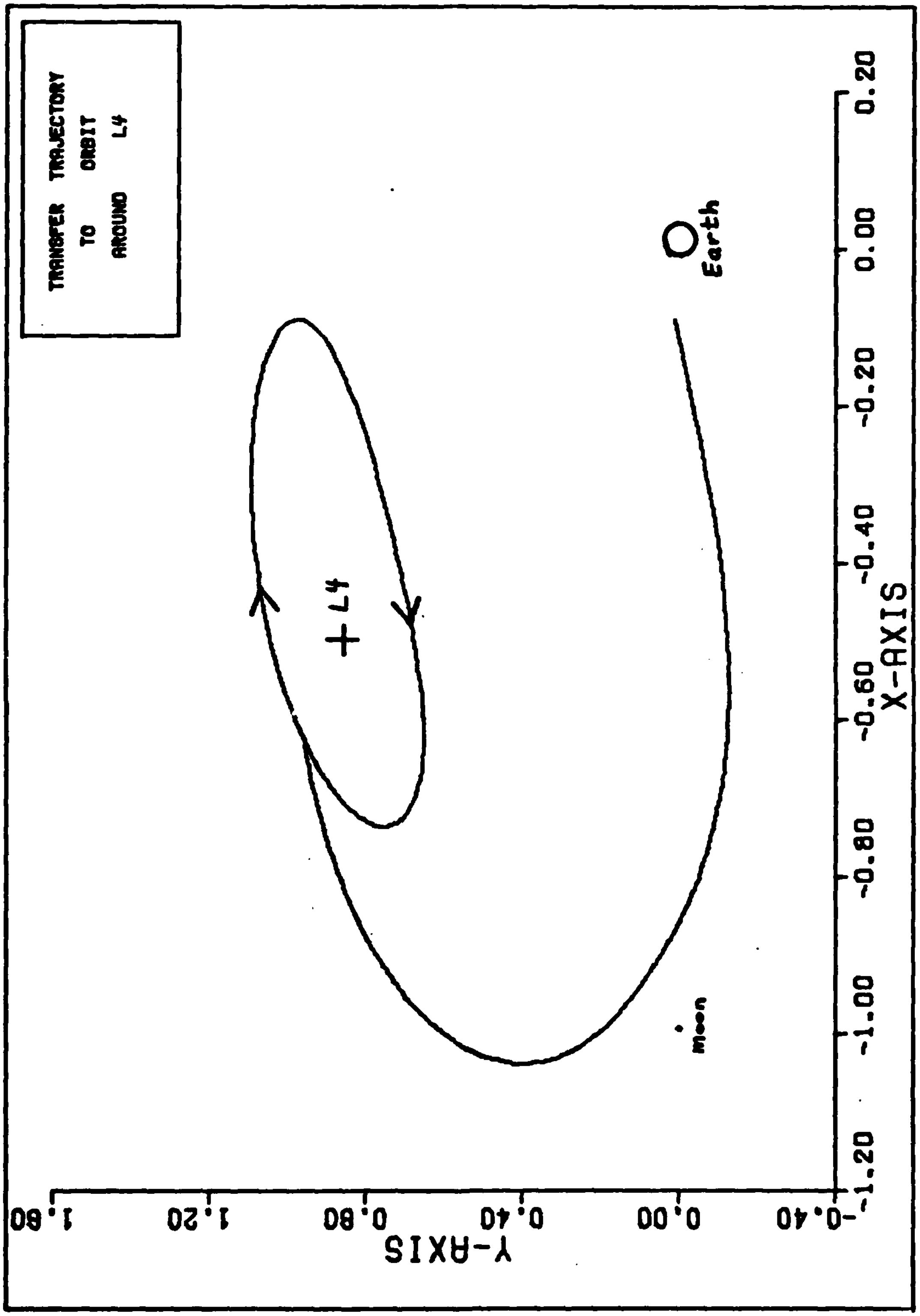
All ten transfer trajectories are shown in Figs (13-22). In these figures the Earth is located at (0.121396054, 0.0) and the Moon at (-0.9878603946, 0.0).

Discussion

If the information found in Table 2 is plotted on curves, it is easier to establish relationships between orbits, or at least to attempt to understand why some of the results are as they are.

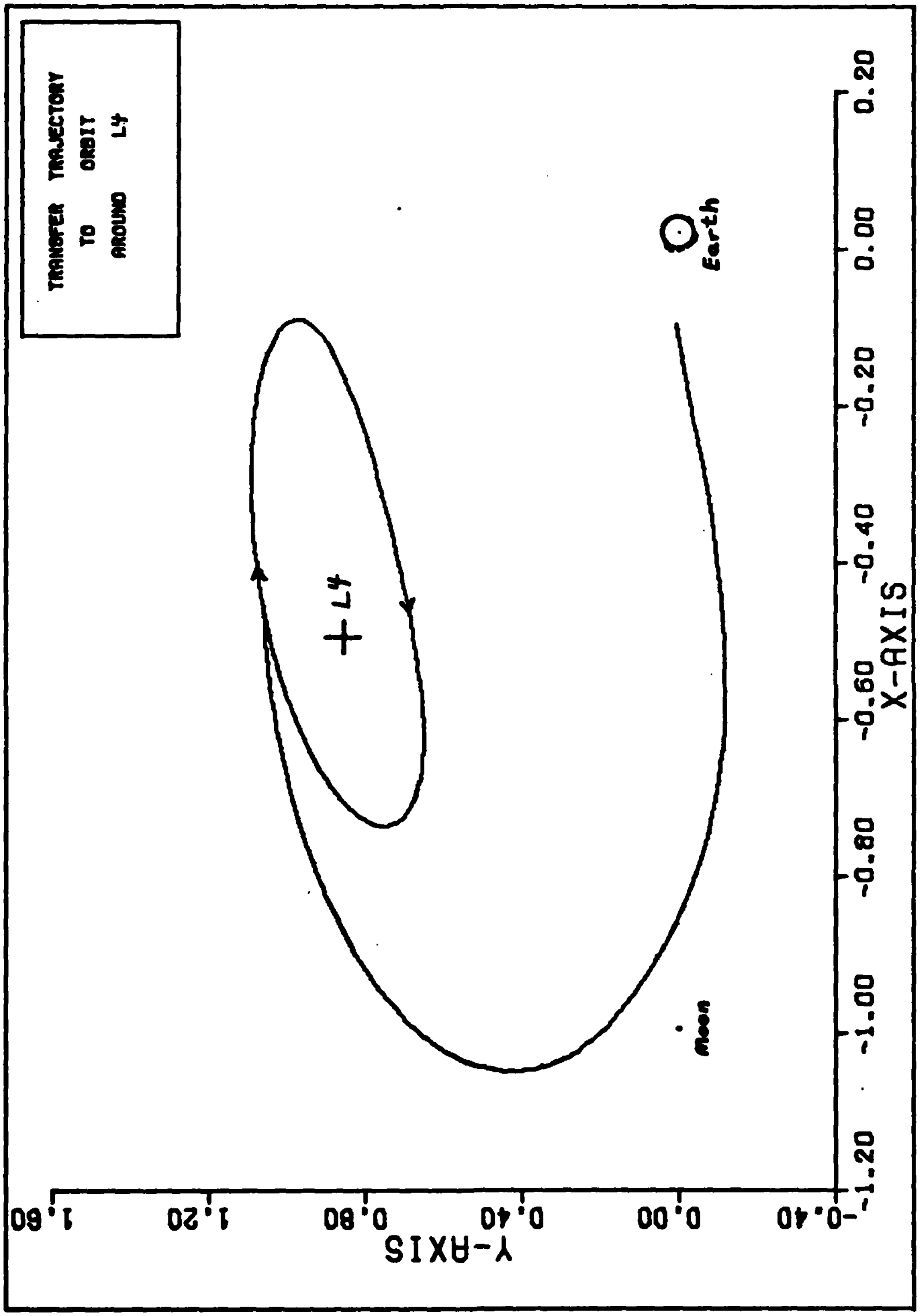
PT	Min ΔV	ϕ	TOF	ΔV_1	ΔV_2
10	3.972123159	.724843243	1.80201	3.305668879	.666454280
20	3.984135344	.708957756	1.963	3.307978640	.676156704
30	4.065693193	.612005368	1.96275	3.312687723	.753005470
40	4.129065109	-.0367607941	1.325341797	3.310239983	.818825126
50	4.196922851	-.246843814	1.15465	3.301168434	.895754417
60	4.285119893	-.341580959	.930834961	3.288204444	.996915449
70	4.315312262	-.190644993	.784493555	3.283408104	1.031904158
80	4.242800165	.154078831	.904406270	3.292165517	.950634648
90	4.129583974	.315731808	1.000114258	3.303329468	.826254506
100	4.025413280	.717849686	1.62143601	3.303352078	.722061203

Table 2. Transfer Trajectory Characteristics



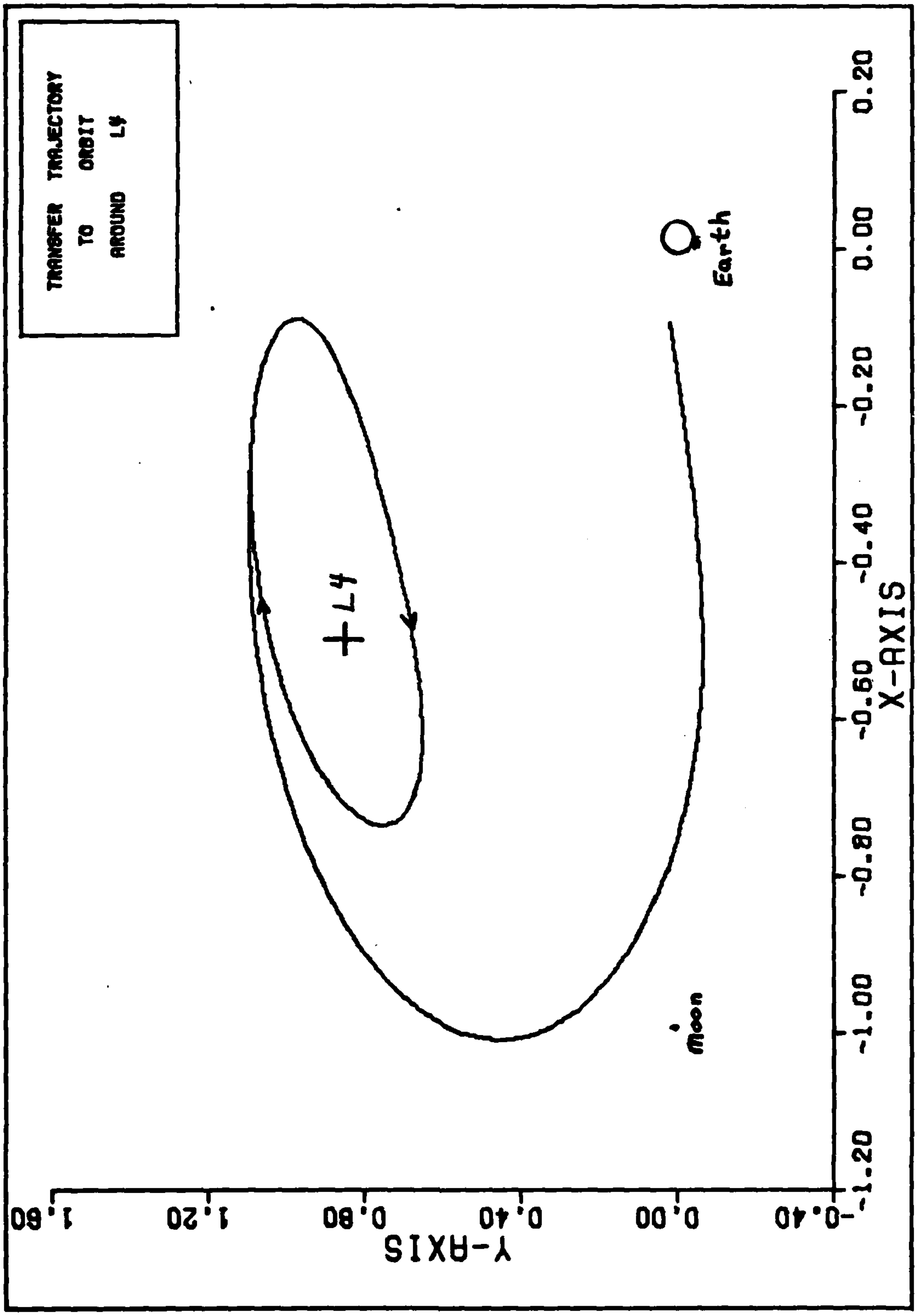
Minimum DeltaV Transfer To Point 10(T=.6283185307)

Fig 13.



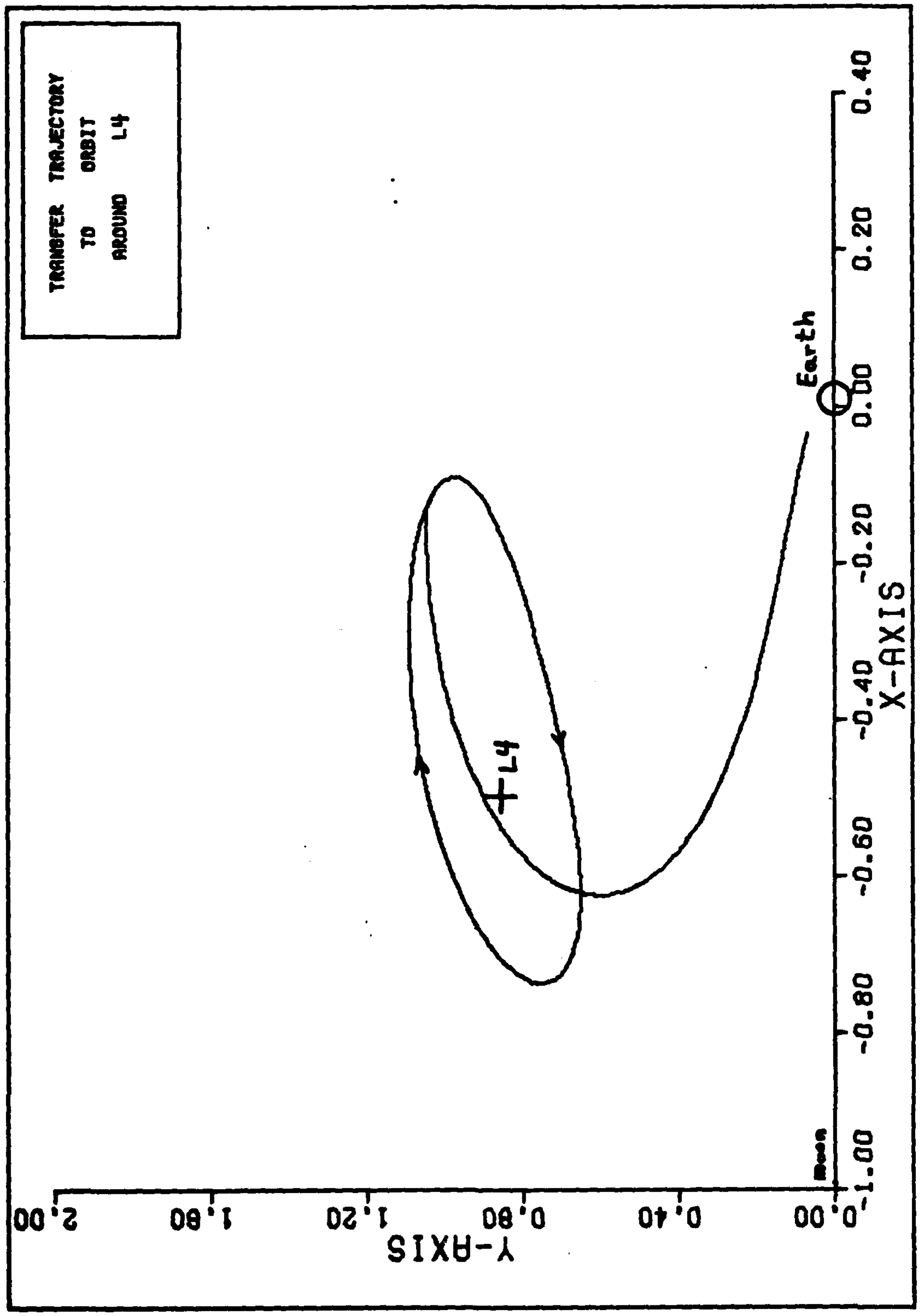
Minimum DeltaV Transfer To Point 20(T=1.2566637061)

Fig 14.



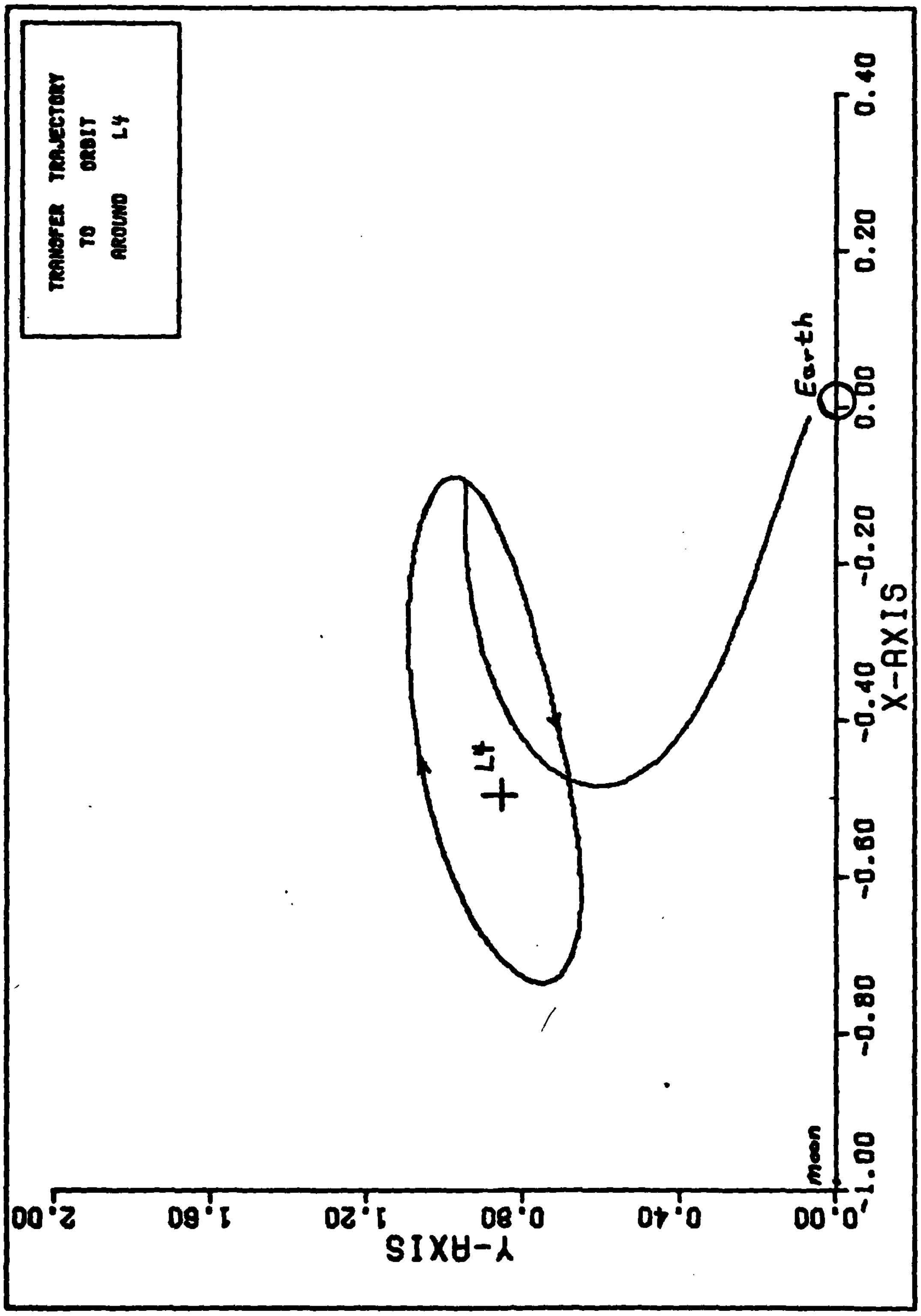
Minimum DeltaV Transfer To Point 30(T=1.884955592)

Fig 15.



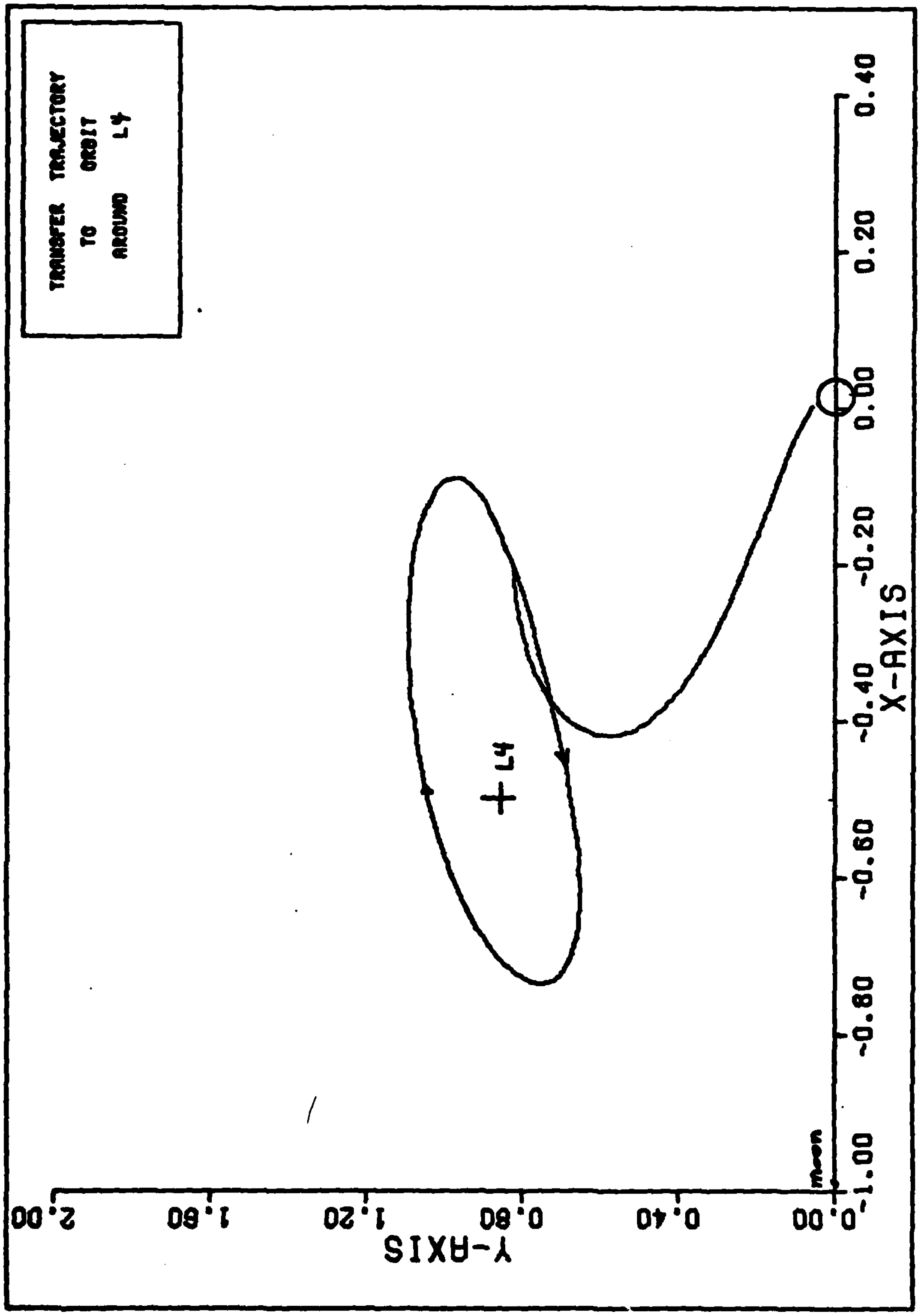
Minimum DeltaV Transfer To Point 40(T=2.513274122)

Fig 16.



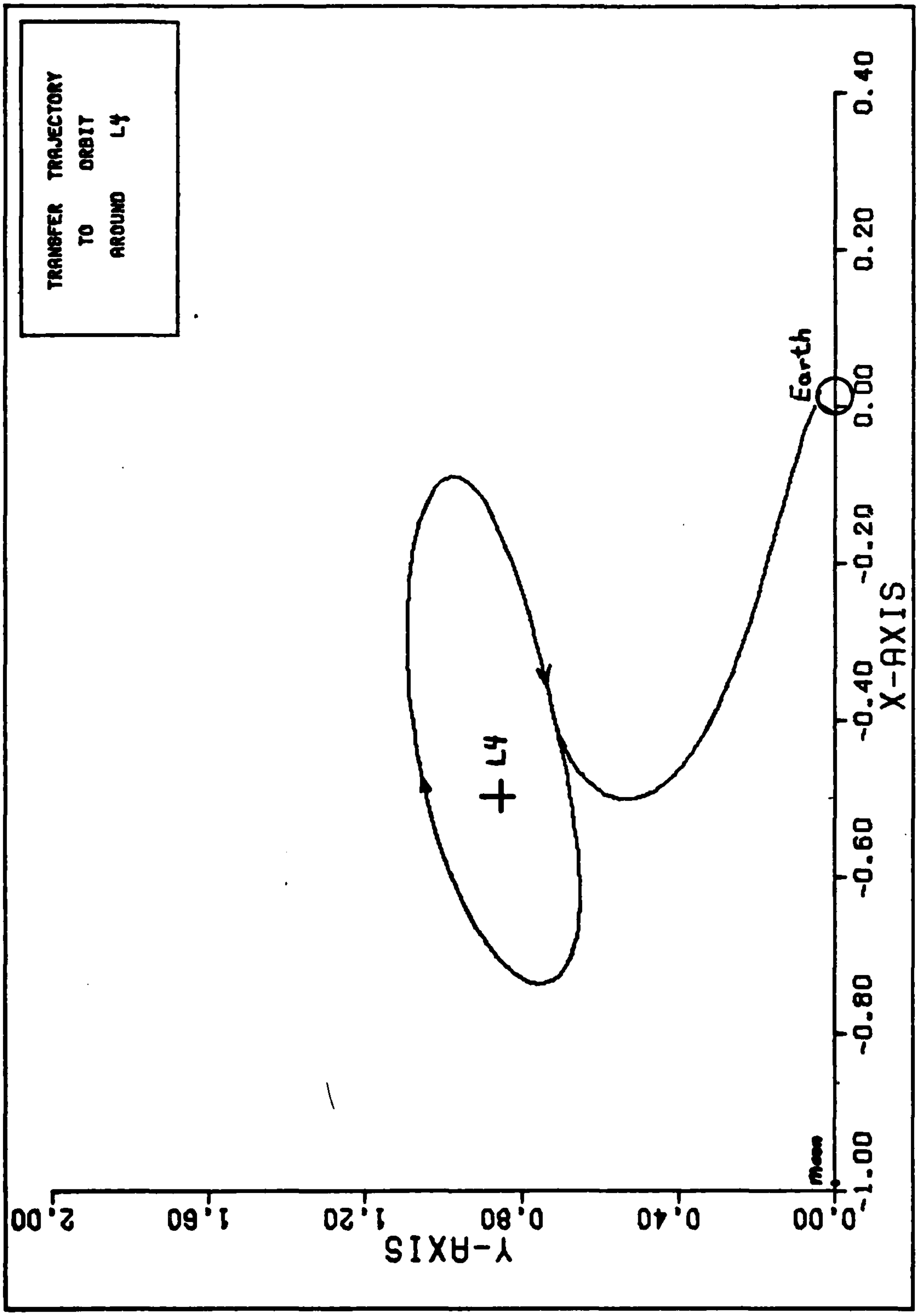
Minimum DeltaV Transfer To Point 50(T=3.14159265)

Fig 17.



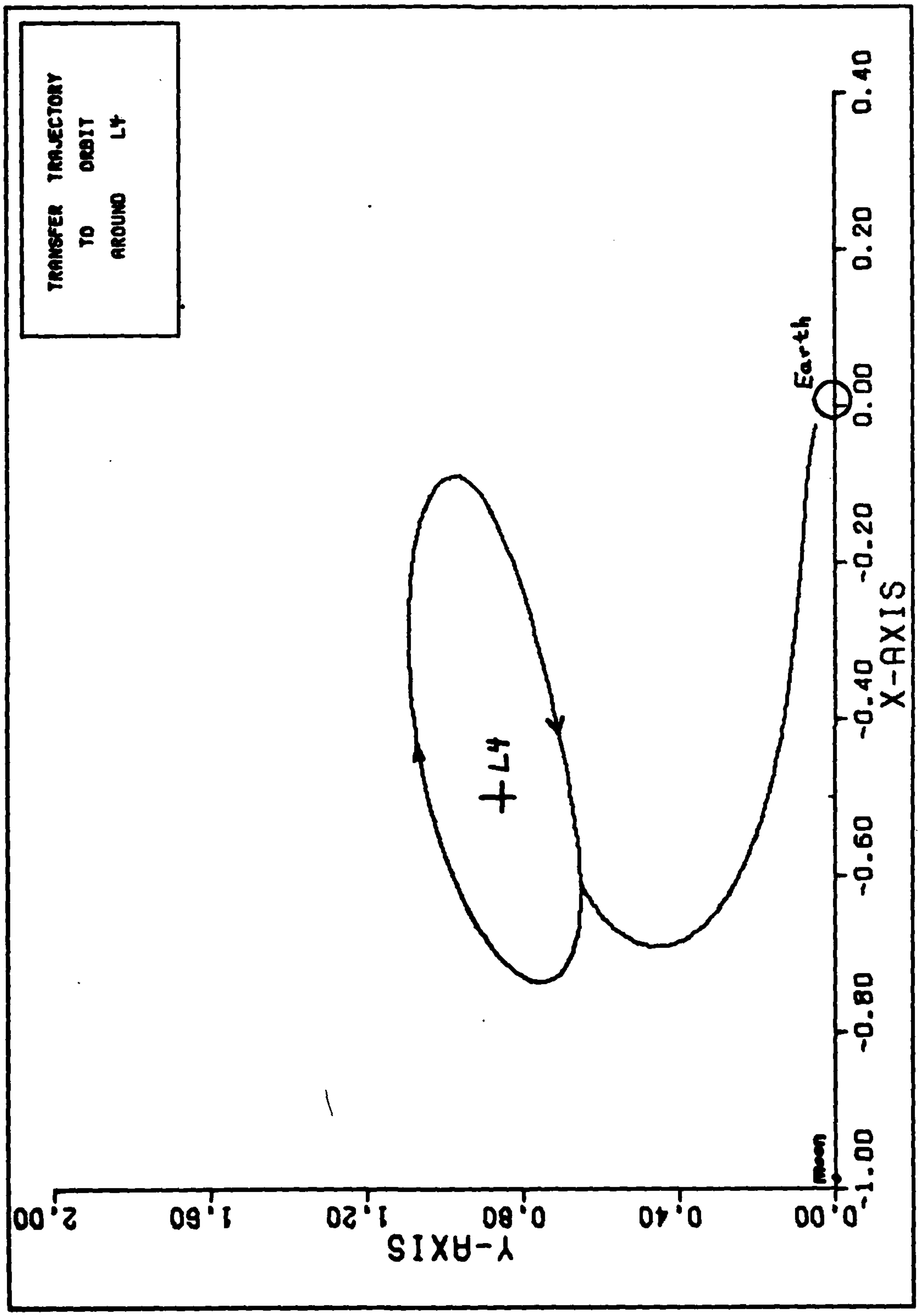
Minimum DeltaV Transfer To Point 60(T=3.76991118)

Fig 18.



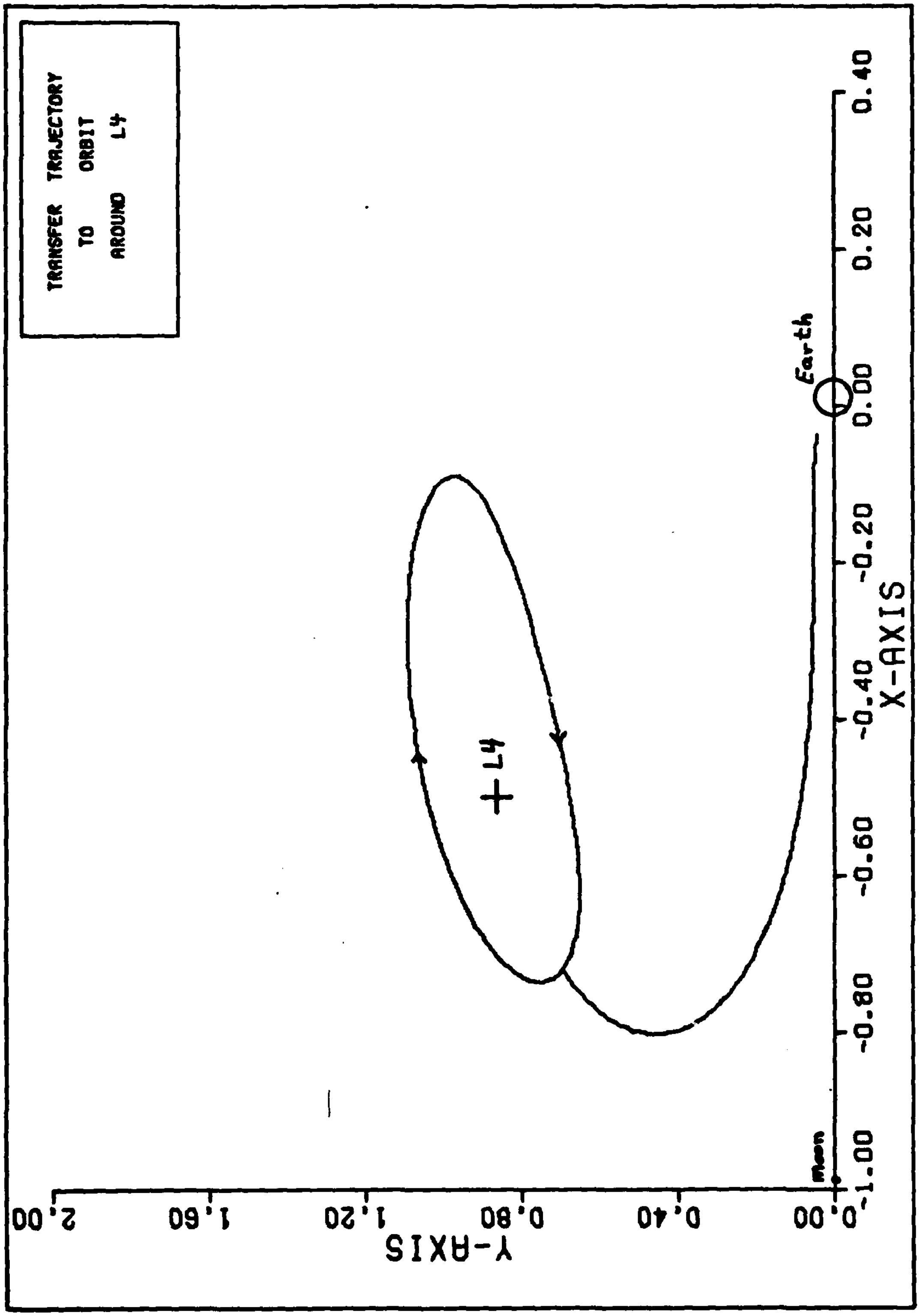
Minimum DeltaV Transfer To Point 70(T=4.398229714)

Fig 19.



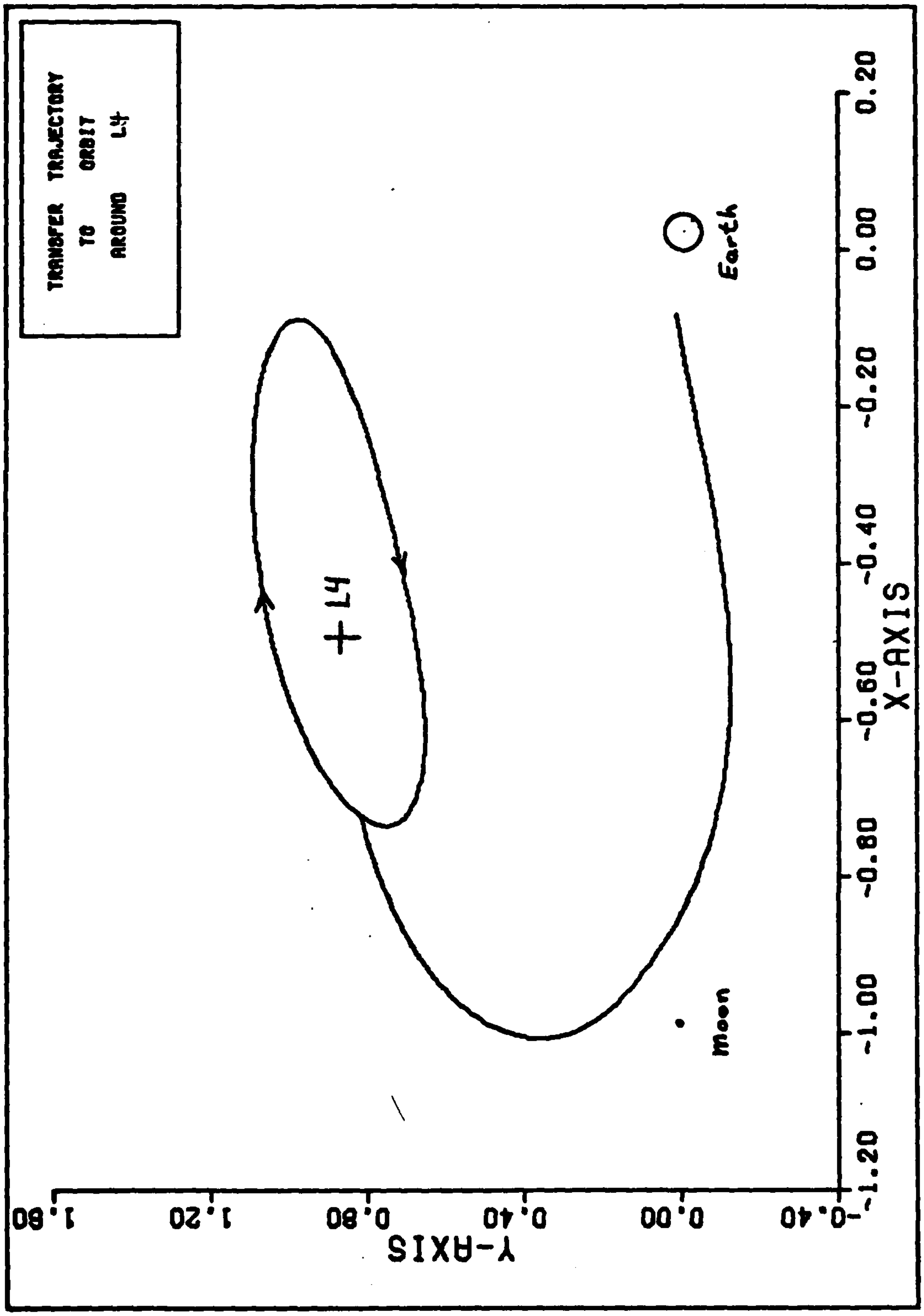
Minimum DeltaV Transfer To Point 80(T=5.026548245)

Fig 20.



Minimum DeltaV Transfer To Point 90(T=5.654866776)

Fig 21.



Minimum DeltaV Transfer To Point 100(T=6.2831853)

Fig 22.

Looking first at the plot of total ΔV versus the target point (Fig 23) and comparing it to plots of its components, ΔV_1, and ΔV_2 (Figs 24, 25), it is possible to see which of the two burns was most influencial in determining the minimum total ΔV. As can be seen, the total ΔV curve is in nearly perfect phase agreement with the ΔV_2 curve. ΔV_1, however, is almost 180° out of phase. This would indicate that ΔV_2 was predominant in determining the shape of the total ΔV curve, even though the ΔV_2 only ranges from 20-27 percent of ΔV_1 in all cases. This can be explained by looking at Wheeler's periodic orbit and examining the target points (Fig 10). Those points requiring the largest ΔV_1, points 30 and 40, are the farthest from the Earth, while those with the lowest ΔV_1, points 60 and 70, are the closest. However, because of the clockwise motion of Wheeler's orbit, the velocity required to enter the stable orbit on the Earth side is nearly opposite that of the arriving satellite and a large ΔV_2 is necessary. In contrast, the velocities at the more distant points on the far side of the orbit are more closely aligned with the incoming satellite's velocity and a smaller ΔV_2 results. This explaines why ΔV_1 and ΔV_2 are out of phase so dramatically and demonstrates the predominance of ΔV_2 in shaping the total ΔV curve.

If Wheeler's orbit was retrograde, or a stable retrograde orbit was found nearby, the total ΔV required could be reduced even more as ΔV_2 for the nearby points would be substantially less due to velocity alignment.

Since the resulting minimum ΔV transfer trajectory has a small dependence on ΔV_1, the choice of parking orbit is essentially an independent one. In other words, the parking orbit has very little effect on what transfer will have the minimum ΔV.

The amplitude of the total ΔV curve (Fig 23) is approximately 330 m/sec. This is a substantial savings and justifies the need for a minimum ΔV transfer. The savings is even more dramatic considering that most of the 330 m/sec savings is in ΔV_2 and fuel for ΔV_2 is payload for ΔV_1.

The TOF curve is very similar to the ΔV_1 curve. This is due primarily to the distance to the target points. The farther distances require a longer time even though a greater initial velocity was used to depart the Earth.

While determining the minimum ΔV transfer to each point two different types of minimums were encountered. One type was a true minimum, in that, ΔV increased as you moved away from the minimum value. The transfer trajectory to point 60 is an example of this type. (See Table 3.)

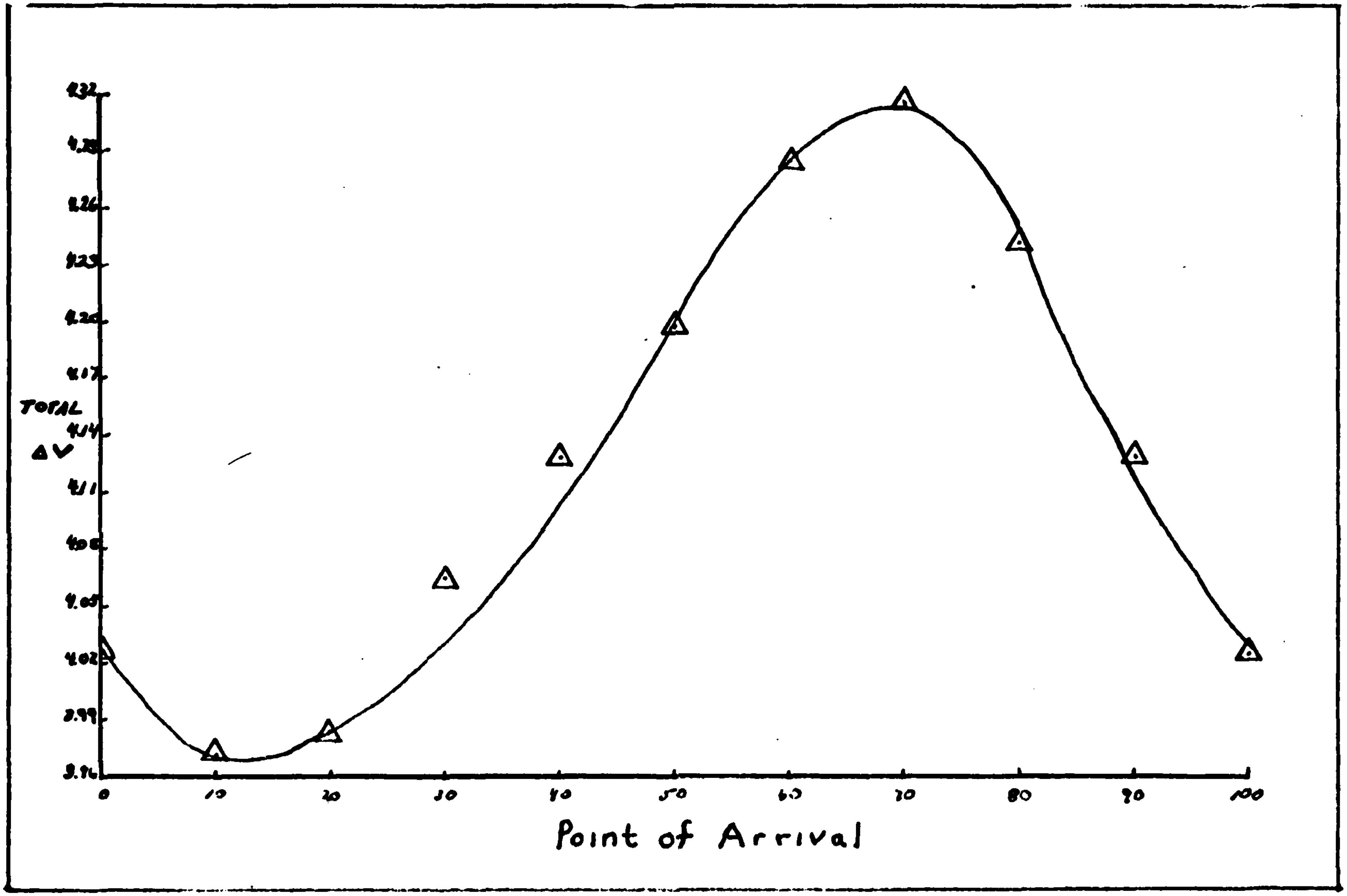


Fig 23. Total ΔV Versus Point Of Arrival

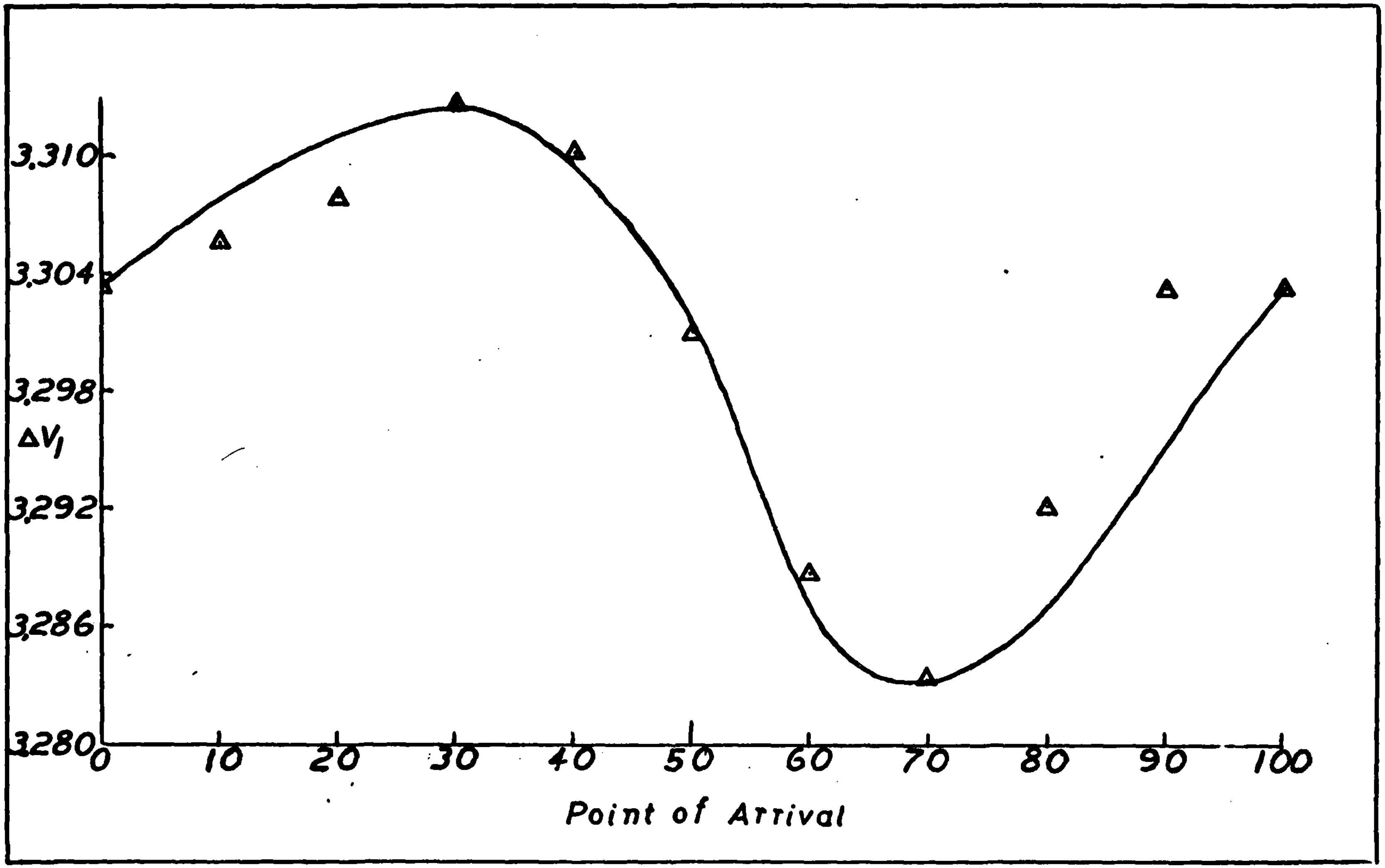


Fig 24. ΔV_1 Versus Point Of Arrival

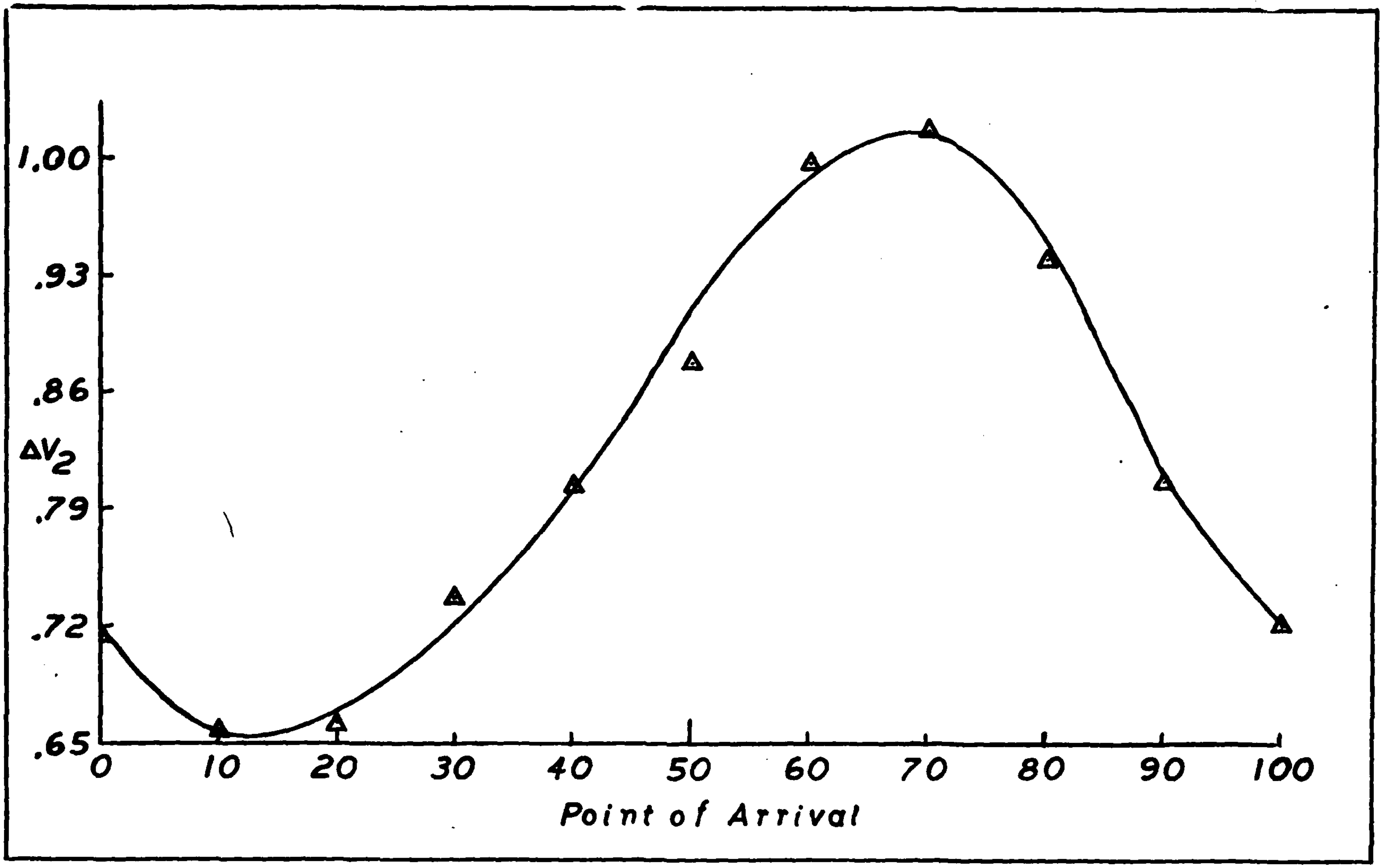


Fig 25. ΔV_2 Versus Point Of Arrival

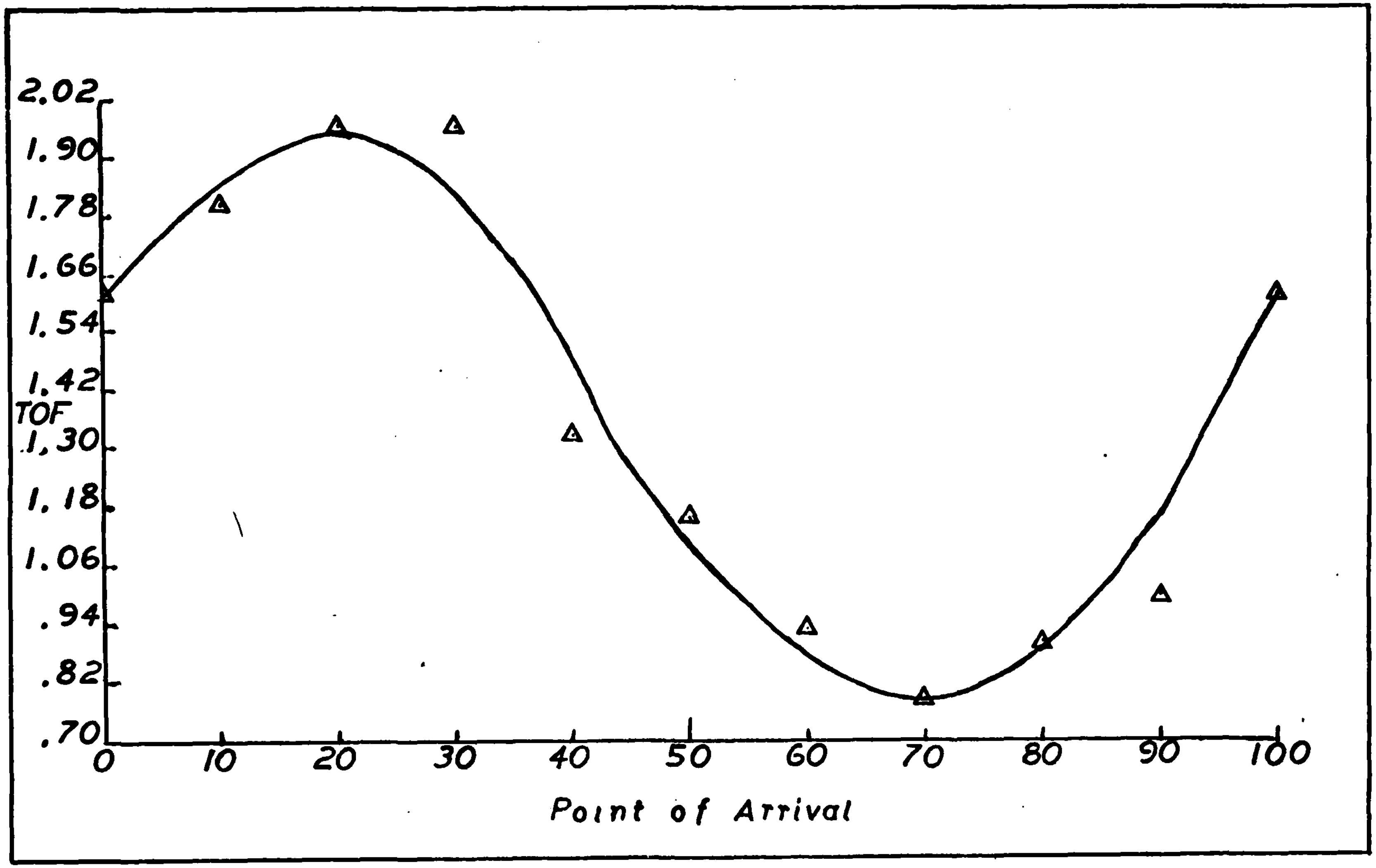


Fig 26. Time Of Flight Versus Point Of Arrival

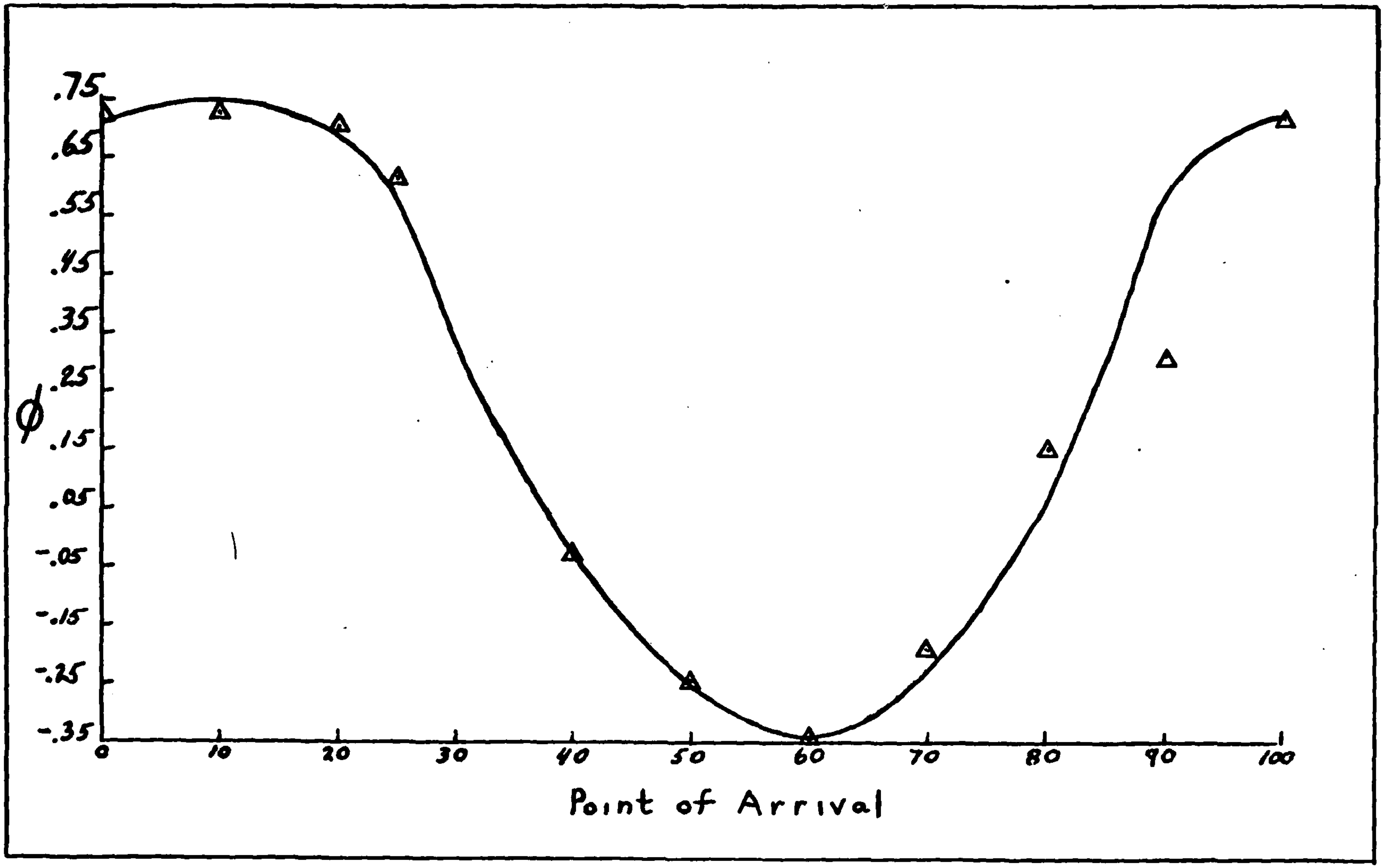


Fig 27. Launch Angle, ϕ, Versus Point Of Arrival

The other type of minimum was one that approached a limiting value. If a value past the minimum was utilized it would not have sufficient velocity to reach the target point. This type is represented by the minimum transfer trajectory to point 10. (See Table 4)

Looking at the results from the transfer to point 60, (Figs 28-31), more indications for the dominance of ΔV_2 can be seen. Total ΔV (Fig 28) and ΔV_2 (Fig 29) are nearly identical in shape while contrasting sharply with ΔV_1 (Fig 30). This supports the conclusion that the second burn, ΔV_2, is dominant in determining the minimum ΔV.

The curves for point 10 are dramatically different than those of point 60. (See Figs 32-35) They do not appear as they would even reach a minimum, even past the limiting values for ΔV_1. All of the ΔV curves show an increasing negative gradient.

One of the results of this thesis is the algorithm that produces the transfer trajectory. However, it is not a very efficient technique and it does not flow smoothly or easily. Too many educated adjustments were required because the B matrix was nearly singular or the required accuracy could not be achieved. For these two reasons a very good initial guess is needed to start the algorithm. Without a good initial guess the path to a solution is very tedious and time consuming.

Point 60

TOF	Min ΔV	ϕ	ΔV_1	ΔV_2
.8	4.30347	-.44728	3.28772	1.01575
.9	4.285376600	-.367219550	3.287848555	.997528
.925	4.28515	-.34646	3.28813	.99702
.9296875	4.285120956	-.342541796	3.288189248	.996931708
.930734961	4.285119901	-.341664715	3.288203114	.996916787
.930834961	4.285119893	-.341580959	3.288204444	.996915449
.930934961	4.285119901	-.341497199	3.288205775	.996914126
.9375	4.285146717	-.335990148	3.288294776	.996851941
.95	4.28540	-.32546	3.28848	.99692
1.000	4.28846	-.28282	3.28934	.99912

Table 3. Data For Point 60

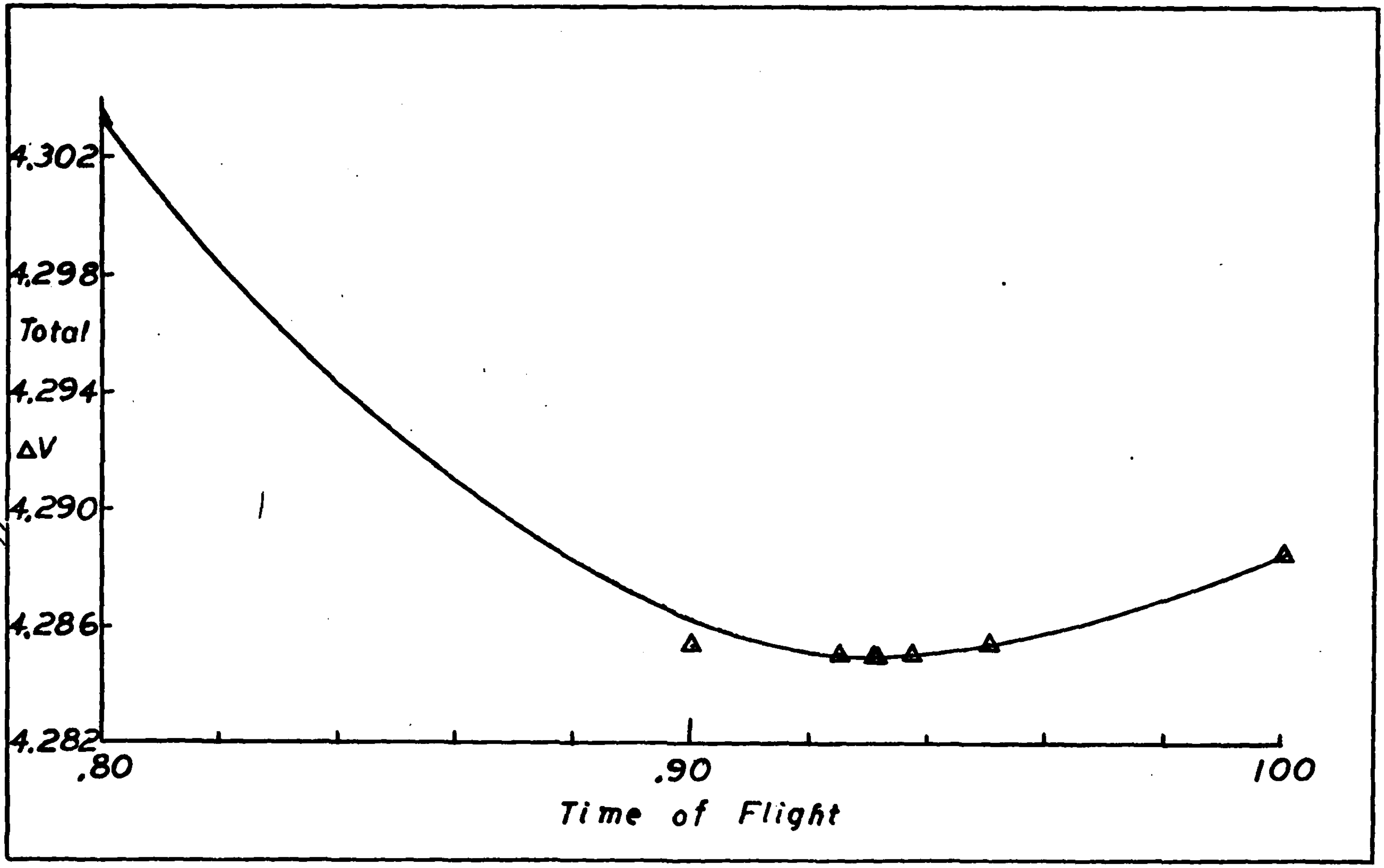


Fig 28 Total Delta V Versus Time of Flight, Point 60

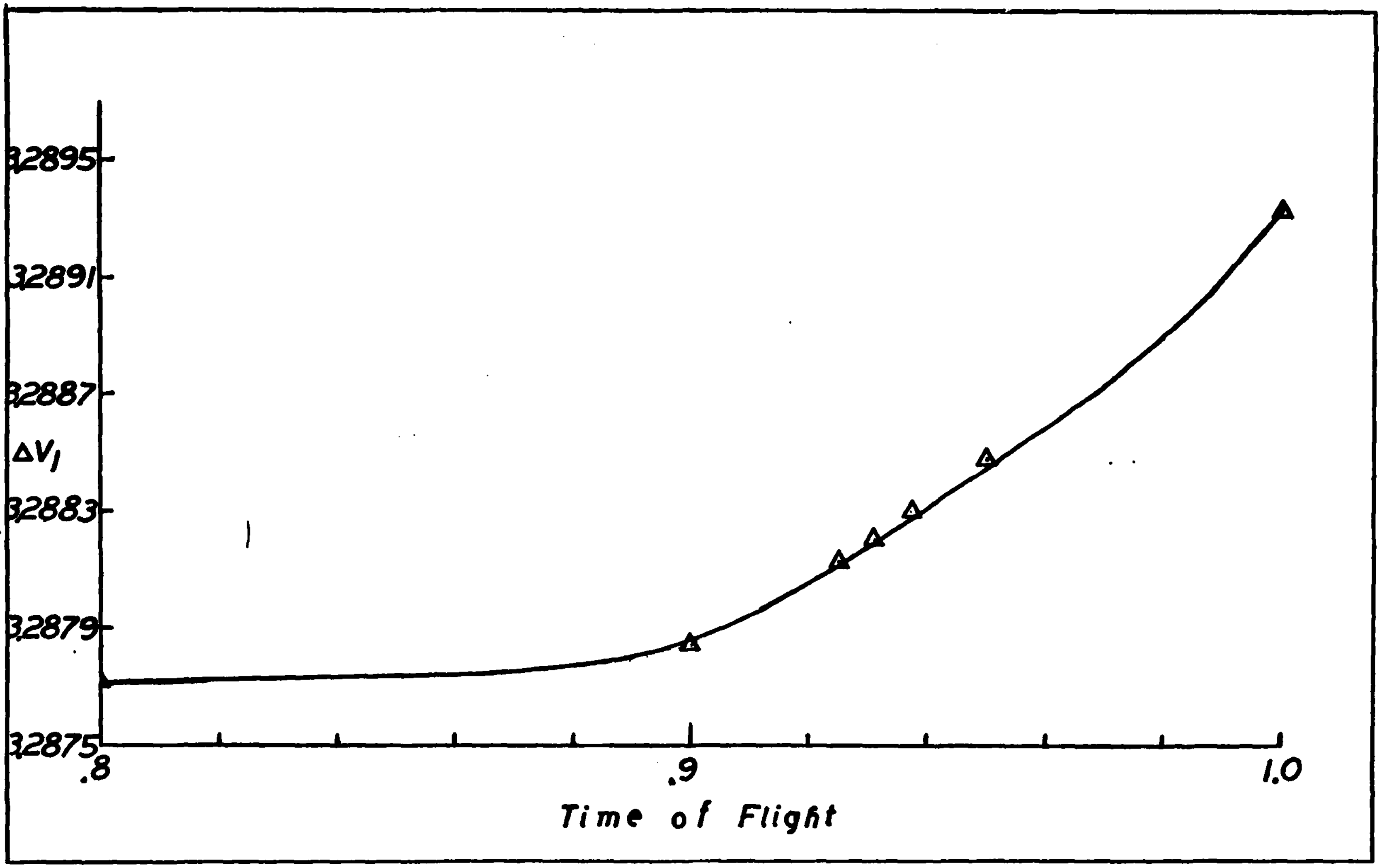


Fig 29 Delta V_1 Versus Time of Flight, Point 60

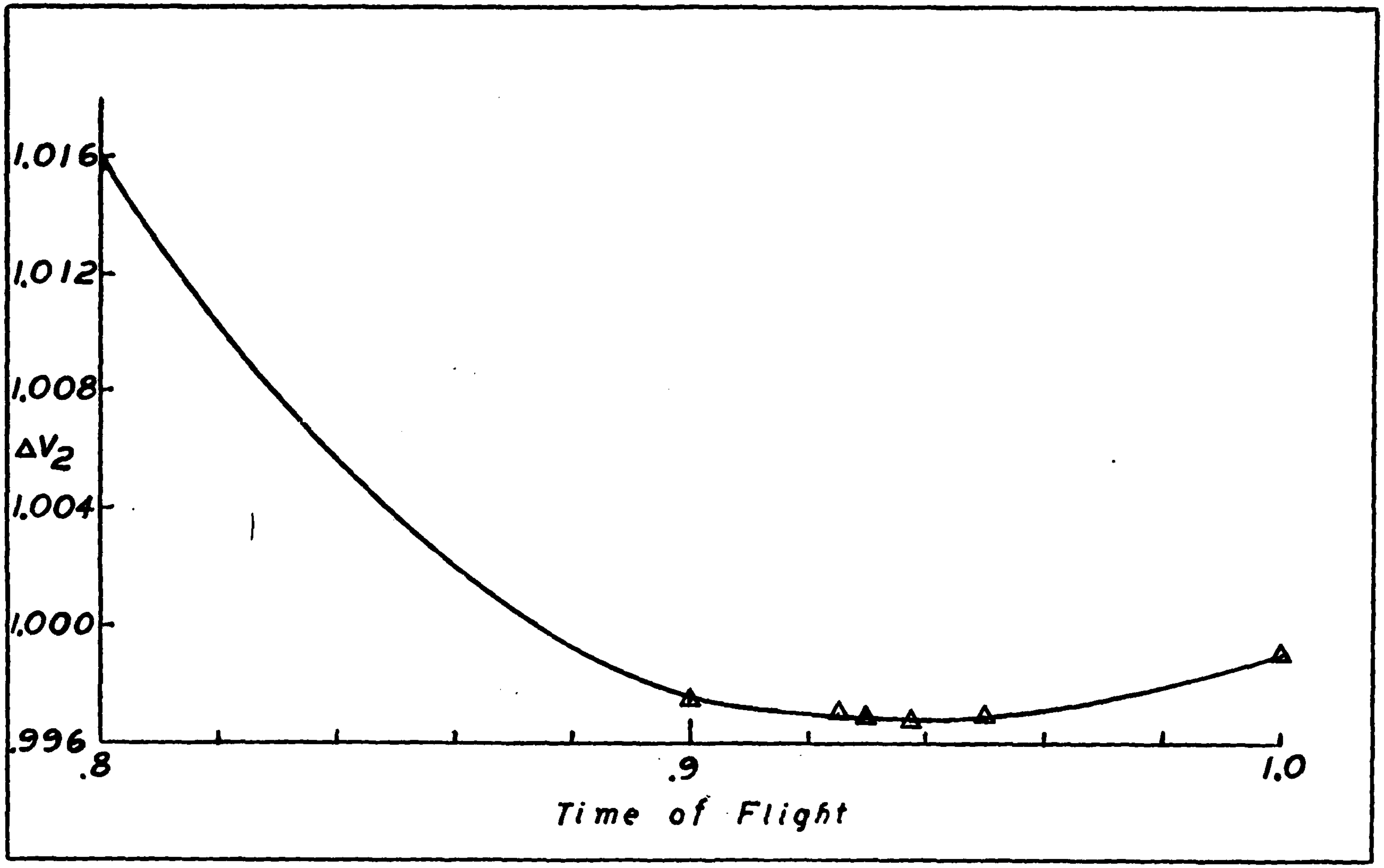


Fig 30 Delta V_2 Versus Time of Flight, Point 60

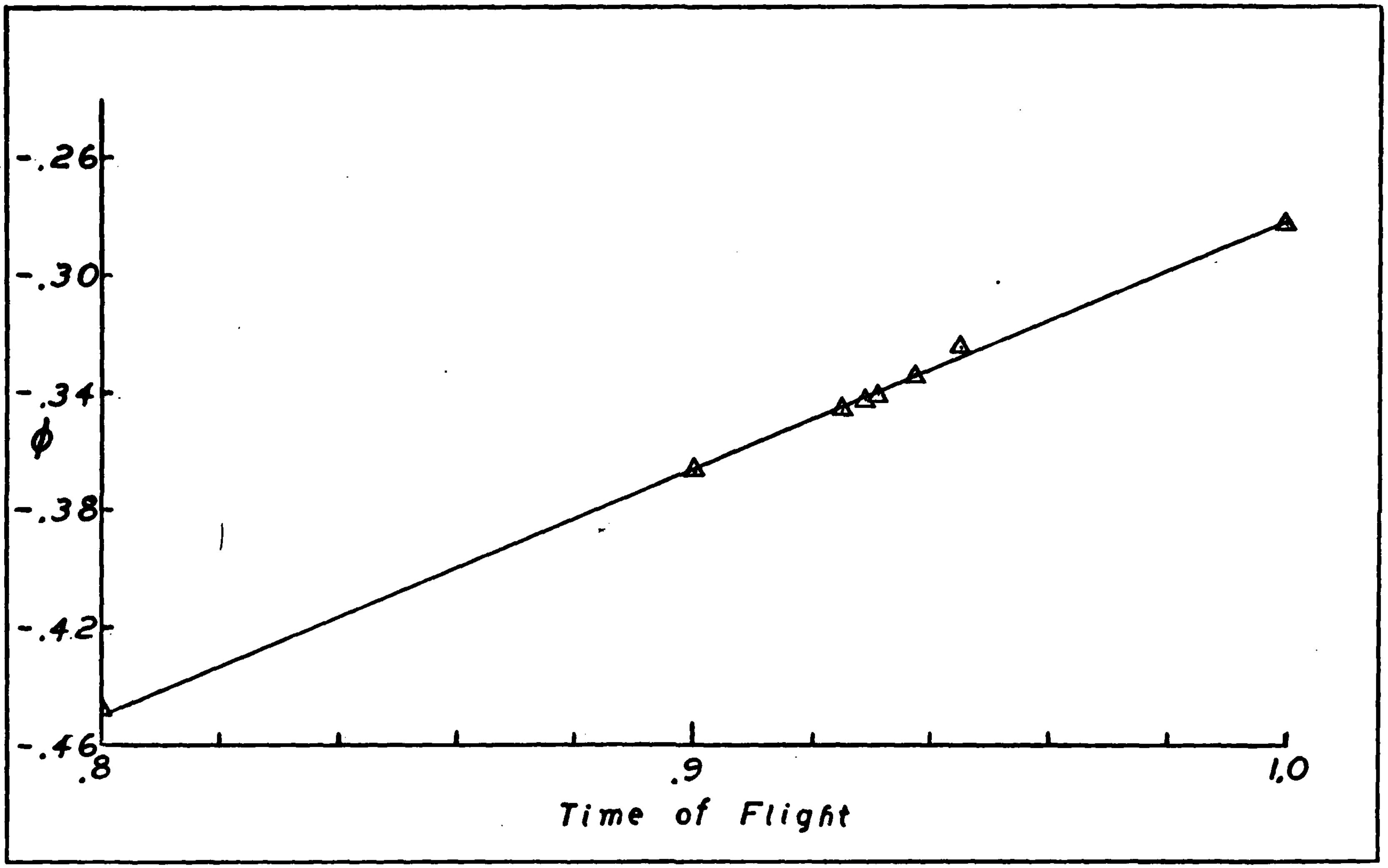


Fig 31 Launch Angle Versus Time of Flight, Point 60

Point 10

TOF	Total ΔV	ϕ	ΔV_1	ΔV_2
1.3	4.038950633	.384679708224	3.313213434	.725737199
1.5	4.022958139	.538278199	3.311048796	.711909343
1.575	4.014838683	.590498723	3.310031328	.704807355
1.65	4.003734325	.638744276	3.308790172	.694944153
1.80	3.972600616	.723784868	3.305713474	.666887142
1.801	3.972363205	.724311612	3.305691292	.666671913
1.802	3.972125537	.724837981	3.305669101	.666456436
1.80201	3.972123159	.724843243	3.305668879	.666454280

Table 4. Data For Point 10

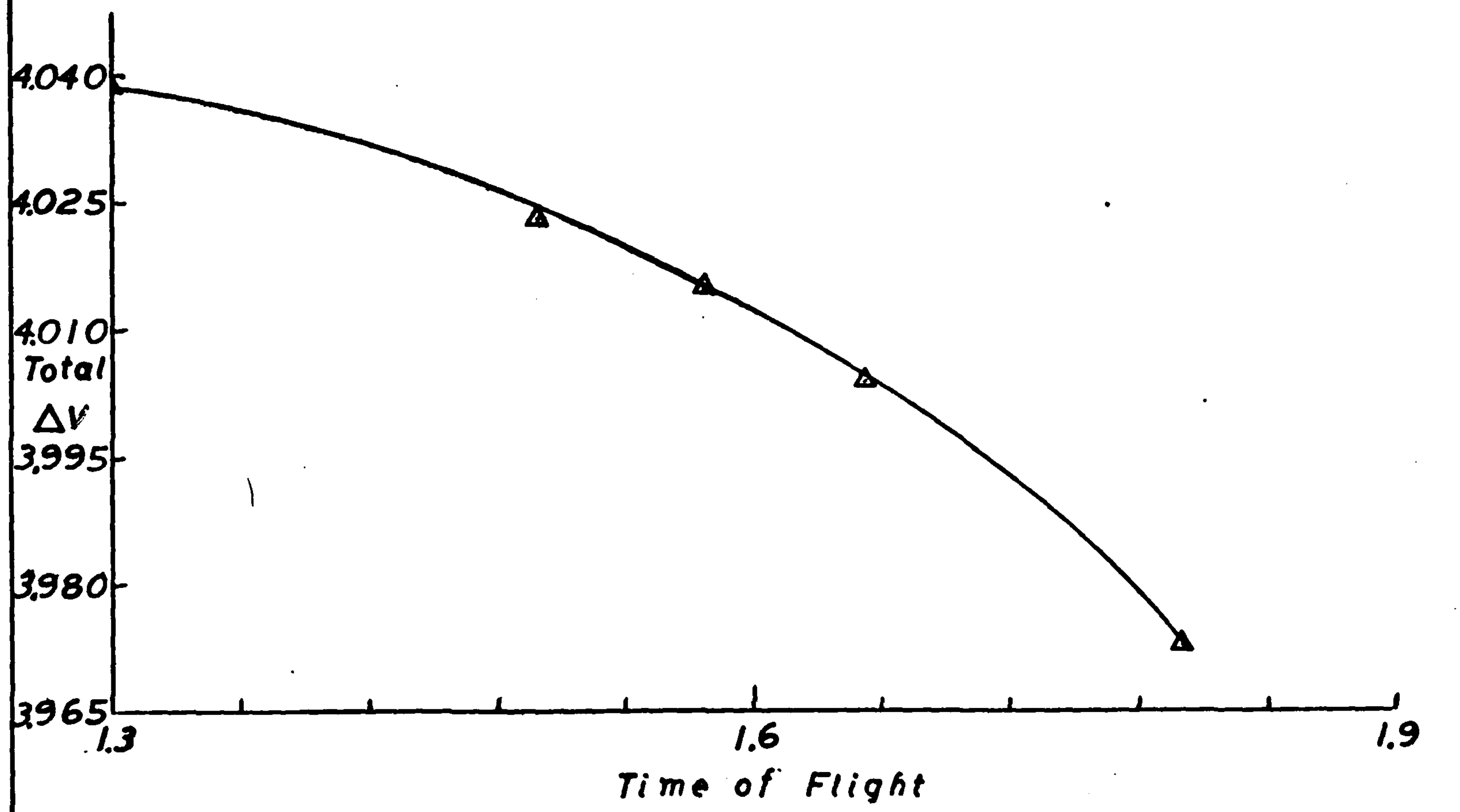


Fig 32 Total Delta V Versus Time of Flight, Point 10

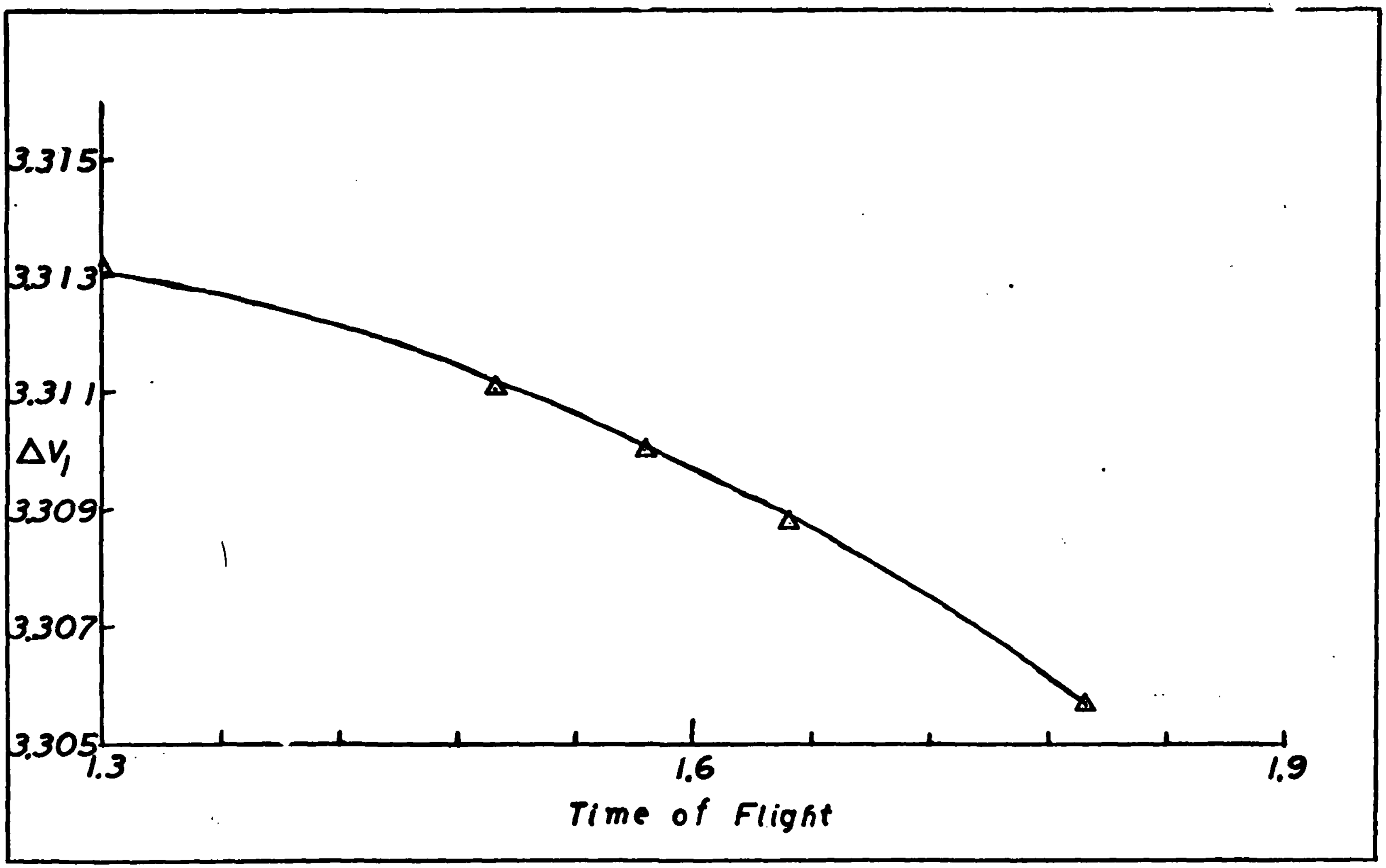


Fig 33 Delta V_1 Versus Time of Flight, Point 10

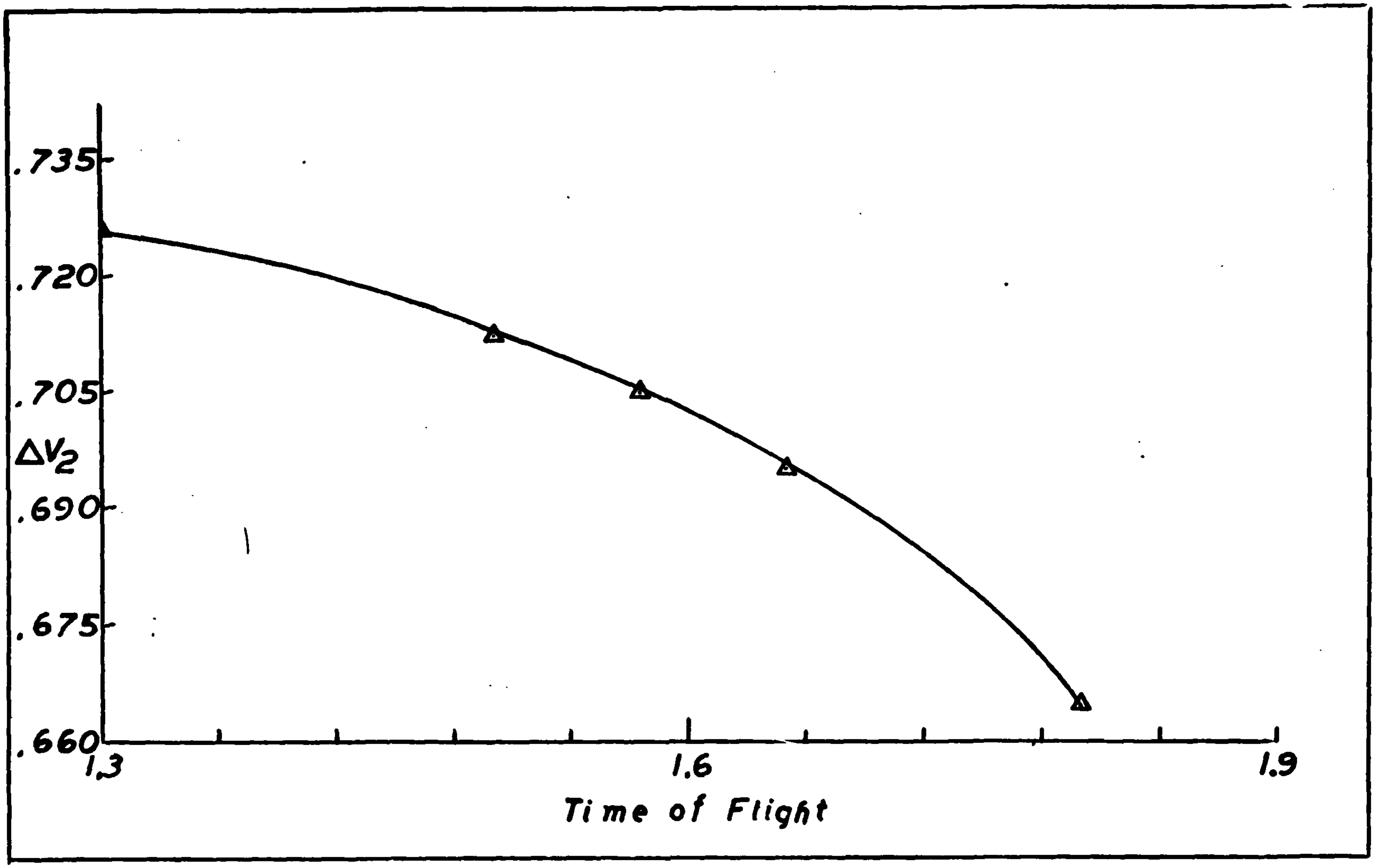


Fig 34 Delta V_2 Versus Time of Flight, Point 10

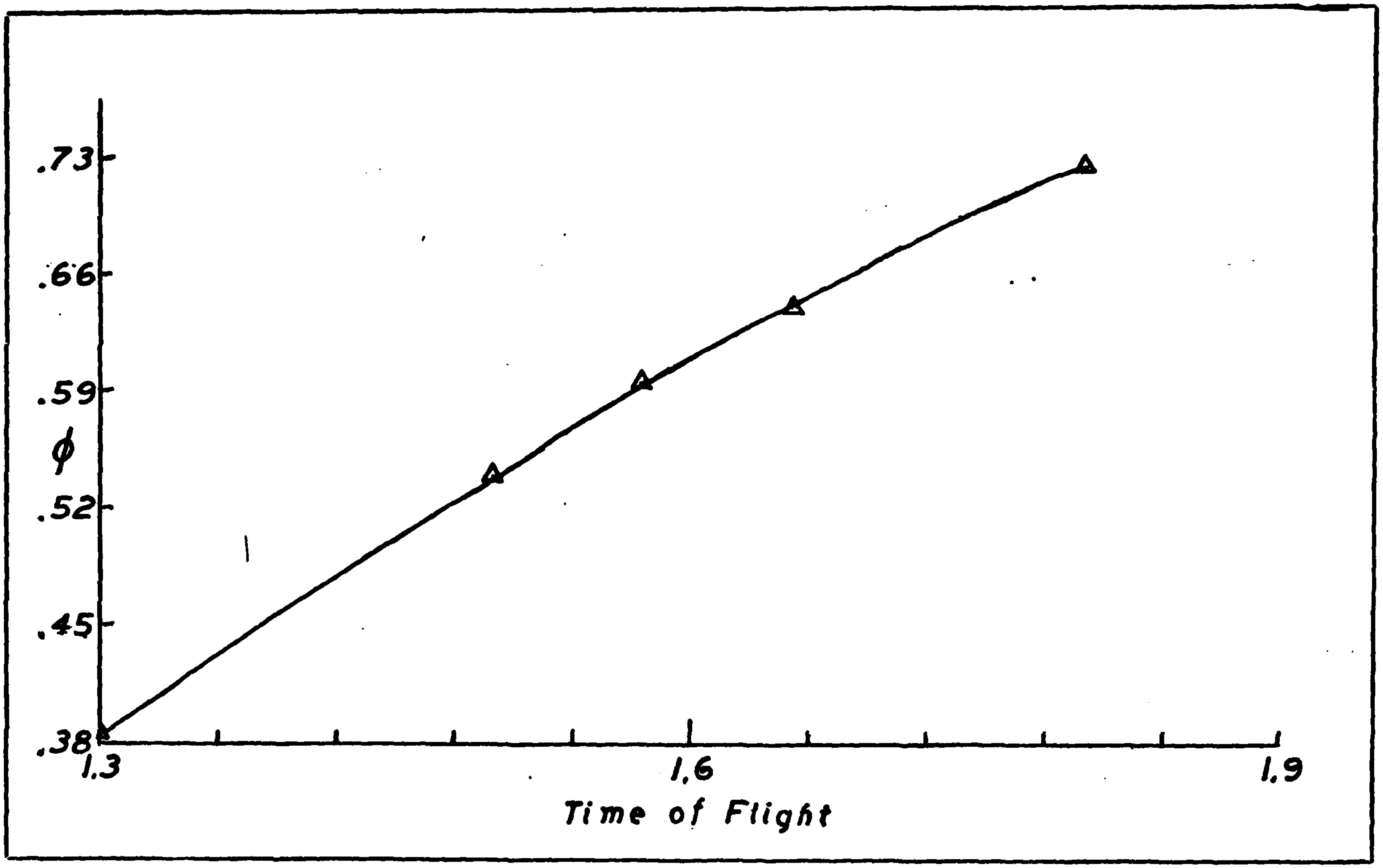


Fig 35 Launch Angle Versus Time of Flight

It is obvious from Fig 23 that the minimum ΔV transfer to Wheeler's orbit does not exist to point 10. Assuming that the total ΔV curve acts like a quadratic near its minimum, a quadratic fit can be done using points 0, 10 and 20 to approximate the minimum value. An equation of the form

$$\Delta V = A\ (pt)^2 + B\ (pt) + C \tag{58}$$

would result. Solving the following three equations will find the coefficient for Eq. (58)

$$4.025413280 = 0^2A + 0B + C \text{ (Point 0)} \tag{59}$$

$$3.972123159 = 100A + 10B + C \text{ (Point 10)} \tag{60}$$

$$3.984135344 = 400A + 20B + C \text{ (Point 20)} \tag{61}$$

Solving these and placing the coefficients in Eq. (58) results in

$$\Delta V = .000326464\ (Pt)^2 - .0086031924\ (Pt) + 4.025413280 \tag{62}$$

To find a minimum

$$\frac{d(\Delta)}{d(Pt)} = .000652928\ Pt - .0086031924 = 0 \tag{63}$$

$$Pt = 13.156$$

This is approximately point 13 of Wheeler's orbit or when arrival time equals .816814090 Tu. Placing 13 in Eq. (62) gives

$$\Delta V = 3.968828695 \text{ Du/Tu} = 3756.98760 \text{ m/sec}$$

This point was not examined by this thesis.

V Summary and Recommendations

Summary

A transfer trajectory with a minimum ΔV was found to Wheeler's orbit. It arrived at Wheeler's orbit at point 10 or when $t = .62831853070$ Tu. The launch parameters for this orbit were also established. From the examination of several orbits it was demonstrated that the minimum ΔV curve for each orbit was very dependent upon ΔV_2. One of the major problems encountered was that some of the transfer trajectories did not have a minimum ΔV, but instead had a limiting value below which the target point could not be reached. Because of this problem the original algorithm to find a minimum ΔV transfer trajectory had to be amended. The resulting algorithm requires a good initial guess for it to operate smoothly.

Recommendations

The primary problem with the algorithm established in this thesis is its inability to handle a singular B matrix. The development of a technique that deals with a singular B matrix would increase the time efficiency of calculating a minimum ΔV orbit.

The transfer trajectory found is not the one with the lowest ΔV possible. In the vicinity of target 10 there will be a trajectory with a lower ΔV. This point was approximated as point 13. Further searching in this area would give a more definite minimum ΔV trajectory.

The results from this thesis indicate that a retrograde orbit that is near Wheeler's would be very attractive in minimizing the cost of a transfer to it. The establishment of a space colony or military structure in a retrograde orbit would substantially lessen the transfer costs from the Earth. Further research for the existance of a retrograde orbit could prove to be very beneficial.

The techniques used in this report should also be applicable if the originating position was other than a low Earth orbit. Other possible launch positions could be on the Moon's surface or that of a large asteroid. Both are possible sources for raw materials for a space colony or military facility. Although the results would be somewhat different, the three launch locations could be combined to form a complete minimum transfer package to Wheeler's orbit or any other desired location.

Bibliography

1. Baker, Robert M. L. Jr. Astrodynamics: Applications and Advanced Topics. New York: Academic Press, Inc. 1967.

2. Bate, R. B., et al. Fundamentals of Astrodynamics, New York: Dover Publications, Inc., 1971.

3. Danby, J. M. A. Fundamentals of Celestrial Mechanics. New York: The Macmillan Company, 1962.

4. Goldberg, L., et al. Annual Review of Astronomy and Astrophysics. Palo Alto, California: Georgia Banta Company, Inc., 1973.

5. Heppenheimer, T. A. "Steps Toward Space Colonization: Colony Location and Transfer Trajectories." Journal of Spacecraft and Rockets, 15:305-12 (1978).

6. Johnson, Richard D., et al. Space Settlements: A Design Study, NASA SP-413, 1977.

7. O'Leary, B., et al. "Trajectory Analysis For Material Transfer From The Moon To A Space Manufacturing Facility." AIAA Journal 57:21-36 (1977).

8. O'Neill, G. The High Frontier. New York: William Morrow and Company, Inc., 1977.

9. Szebehely, V. Theory of Orbits. New York: Academic Press, 1967.

10. Wheeler, J. "Determination of Periodic Orbits And Their Stability In The Very Restricted Four-Body Model In The Vicinity Of The Lagrangian Points L4 and L5." Unpublished thesis. Wright-Patterson Air Force Base, Ohio: Air Force Institute of Technology, December 1978.

Appendix A

Derivation of A Matrix

The A matrix is defined by Eq. (36)

$$A(t) = \left[\frac{\partial \underline{x}_i}{\partial \underline{x}_j}\right]_{\substack{i = 1,\ 4 \\ j = 1,\ 4}} \tag{A-1}$$

A 4 x 4 matrix results from (A-1) with the following components:

$$\frac{\partial \dot{x}_1}{\partial x_1} = 0 \tag{A-2}$$

$$\frac{\partial \dot{x}_1}{\partial x_2} = 0 \tag{A-3}$$

$$\frac{\partial \dot{x}_1}{\partial x_3} = 1.0 \tag{A-4}$$

$$\frac{\partial \dot{x}_1}{\partial x_4} = 0 \tag{A-5}$$

$$\frac{\partial \dot{x}_2}{\partial x_1} = 0 \tag{A-6}$$

$$\frac{\partial \dot{x}_2}{\partial x_2} = 0 \tag{A-7}$$

$$\frac{\partial \dot{x}_2}{\partial x_3} = 0 \tag{A-8}$$

$$\frac{\partial \dot{x}_2}{\partial x_4} = 1.0 \tag{A-9}$$

$$\frac{\partial \dot{x}_3}{\partial x_1} = (\dot{\theta}+\dot{\alpha})^2 - \frac{(1-\mu)G}{r_3^3} - \frac{G\mu}{r_2^3} + \frac{3(1-\mu)(x_1-\mu)^2 G}{r_1^5}$$

$$+\frac{3G(1-\mu+x_1)^2}{r_2^5} - \frac{GM_s}{r_3^3} + \frac{3G(x_1+R_{SE}\cos\,\Theta)^2 M_s}{r_3^5} \tag{A-10}$$

$$\frac{\partial\dot{x}_3}{\partial x_2} = \frac{3Gx_2(x_1-\mu)(1-\mu)}{r_1^5} + \frac{3Gx_2\mu(1-\mu+x_1)}{r_2^5}$$

$$+ \frac{3GM_s(x_2-R_{SE}\sin\Theta)(x_1+R_{SE}\cos\Theta)}{r_3^5} \tag{A-11}$$

$$\frac{\partial\dot{x}_3}{\partial x_3} = 0 \tag{A-12}$$

$$\frac{\partial\dot{x}_3}{\partial x_4} = 2(\dot{\Theta}+\dot{\alpha}) \tag{A-13}$$

$$\frac{\partial\dot{x}_4}{\partial x_1} = \frac{\partial\dot{x}_3}{\partial x_2} \tag{A-14}$$

$$\frac{\partial\dot{x}_4}{\partial x_2} = (\dot{\Theta}+\dot{\alpha}) - \frac{(1-\mu)G}{r_1^3}$$

$$- \frac{\mu G}{r_2^3} + \frac{3G(1-\mu)(x_2)^2}{r_1^5} + \frac{3G\mu x_2^2}{r_2^5}$$

$$- \frac{GM_s}{} + \frac{3GM_s(x_2-R_{SE}\sin\Theta)^2}{r_3^3} \tag{A-15}$$

$$\frac{\partial\dot{x}_4}{\partial x_3} = -2(\dot{\Theta}+\dot{\alpha}) \tag{A-16}$$

$$\frac{\partial\dot{x}_4}{\partial x_4} = 0 \tag{A-17}$$

$\dot{x}_1$, $\dot{x}_2$, $\dot{x}_3$ and $\dot{x}_4$ are the first-order, non-linear equations of motion defined by Eqs. (29) to (32)

The A matrix linearizes small changes in the dynamics along the trajectory and it also propogates Φ along the trajectory. A is organized as follows:

$$A = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} & \frac{\partial \dot{x}_1}{\partial x_3} & \frac{\partial \dot{x}_1}{\partial x_4} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} & \frac{\partial \dot{x}_2}{\partial x_3} & \frac{\partial \dot{x}_2}{\partial x_4} \\ \frac{\partial \dot{x}_3}{\partial x_1} & \frac{\partial \dot{x}_3}{\partial x_2} & \frac{\partial \dot{x}_3}{\partial x_3} & \frac{\partial \dot{x}_3}{\partial x_4} \\ \frac{\partial \dot{x}_4}{\partial x_1} & \frac{\partial \dot{x}_4}{\partial x_2} & \frac{\partial \dot{x}_4}{\partial x_3} & \frac{\partial \dot{x}_4}{\partial x_4} \end{bmatrix} \qquad \text{(A-18)}$$

Appendix B

Derivation of the B Matrix

Utilizing Fig. (4), expressions for the x, y coordinates of the satellite in its staging orbit can be put in terms of the launch angle, ϕ.

$$x = r_1 \cos\phi + \mu \tag{B-1}$$

$$y = r_1 \sin\phi \tag{B-2}$$

From these equations the velocity of the satellite is found by differentiating (B-1) and (B-2).

$$\dot{x} = \dot{r}_1 \cos\phi - r_1 \dot{\phi} \sin\phi + (\omega \times r) \text{ term} \tag{B-3}$$

$$\dot{y} = \dot{r}_1 \sin\phi + r_1 \dot{\phi} \cos\phi + (\omega \times r) \text{ term} \tag{B-4}$$

The $(\omega \times r)$ terms are the result of the $x_e - y_e$ coordinate rotating with respect to the $x_E - y_E$ system which in turn rotates with respect to the $x_I - y_I$ system. The $(\omega \times r)$ term from the $x_E - y_E$ system is on the order of $(r_1 + \mu)$ and is neglected. From the inertial frame the motion of the satellite appears to be a straight line and the $(\omega \times r)$ term is neglected here, also.

AD-A079 881 AIR FORCE INST OF TECH WRIGHT-PATTERSON AFB OH SCHOO--ETC F/G 22/3
DETERMINATION OF THE MINIMUM DELTA V TRANSFER TRAJECTORY FROM A--ETC(U)
DEC 79 T E WEIMER

UNCLASSIFIED AFIT/GA-AA/79D-11 NL

2 OF 2
AD A079881

END
DATE FILMED
2-80
DDC

Realizing that $\dot{r}_1 = 0$ and $r_1\dot{\phi} = v_c$, the circular orbit velocity, Eq. (B-3) and (B-4) become

$$\dot{x} = - v_c \sin \phi \qquad \text{(B-5)}$$

$$\dot{y} = v_c \cos \phi \qquad \text{(B-6)}$$

Adding the impulse velocity change to these terms gives the initial transfer velocity of the satellite at t_0^+ .

$$\dot{x} = - v_c \sin \phi - \Delta v_1 \sin \phi = - (v_c + \Delta v_1) \sin \phi \qquad \text{(B-7)}$$

$$\dot{y} = v_c \cos \phi + \Delta v_1 \cos \phi = (v_c + \Delta v_1) \cos \phi \qquad \text{(B-8)}$$

Eqs. (B-1, 2, 7, 8) provide the initial conditions $x(t_0)$, to begin the integration scheme.

To linearize the two-point boundary value problem Eq. (38) is utilized. It relates errors at t_1 to changes at t_0.

$$\delta x(t_1) = \Phi(t_1, t_0)\ \delta x(t_0) \qquad \text{(38)}$$

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta \dot{x} \\ \delta \dot{y} \end{bmatrix} = \begin{bmatrix} \\ \Phi \\ \\ \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta \dot{x} \\ \delta \dot{y} \end{bmatrix}_{t_0} \qquad \text{(B-9)}$$

However, the initial conditions, $x(t_0)$, are related to the launch parameters ϕ and Δv_1 by Eqs. (B-1, 2, 7, 8). Utilizing Eq. (45)

$$\delta x(t_0) = C\delta L \tag{45}$$

where

$$C = \begin{bmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \Delta v_1} \\ \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \Delta v_1} \\ \frac{\partial \dot{x}}{\partial \phi} & \frac{\partial \dot{x}}{\partial \Delta v_1} \\ \frac{\partial \dot{y}}{\partial \phi} & \frac{\partial \dot{y}}{\partial \Delta v_1} \end{bmatrix} \tag{B-10}$$

so,

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta \dot{x} \\ \delta \dot{y} \end{bmatrix} = \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} \delta \phi \\ \delta \Delta v_1 \end{bmatrix} \tag{B-11}$$

Combining (B-9) and (B-11) produces

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta \dot{x} \\ \delta \dot{y} \end{bmatrix}_t = \begin{bmatrix} \Phi \end{bmatrix} \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} \delta \phi \\ \delta \Delta v_1 \end{bmatrix} \tag{B-12}$$

Since position error is all that we are interested in, all corrections to $\dot{x}$ and $\dot{y}$ at t_1 are ignored. Also, calling the (Φ) (C) product by the label, B, results in

$$\begin{bmatrix} \delta x \\ \delta y \end{bmatrix}_{t_1} = B \begin{bmatrix} \delta \phi \\ \delta \Delta v_1 \end{bmatrix}_{t_0}$$

$$\text{or} \quad \begin{bmatrix} \delta \phi \\ \delta \Delta v_1 \end{bmatrix}_{t_0} = B^{-1} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}_{t_1} \tag{B-13}$$

The B matrix is defined as follows:

$$B_{11} = \Phi_{11}C_{11} + \Phi_{12}C_{21} + \Phi_{13}C_{31} + \Phi_{14}C_{41} \tag{B-14}$$

$$B_{12} = \Phi_{11}C_{12} + \Phi_{12}C_{22} + \Phi_{13}C_{32} + \Phi_{14}C_{42} \tag{B-15}$$

$$B_{21} = \Phi_{21}C_{11} + \Phi_{22}C_{21} + \Phi_{23}C_{31} + \Phi_{24}C_{41} \tag{B-16}$$

$$B_{22} = \Phi_{21}C_{12} + \Phi_{22}C_{22} + \Phi_{23}C_{32} + \Phi_{24}C_{42} \tag{B-17}$$

where

$$C_{11} = \frac{\partial x}{\partial \phi} = -r_1 \sin \phi \tag{B-18}$$

$$C_{12} = \frac{\partial x}{\partial \Delta v_1} = 0 \tag{B-19}$$

$$C_{21} = \frac{\partial y}{\partial \phi} = r_1 \cos \phi \tag{B-20}$$

$$C_{22} = \frac{\partial y}{\partial \Delta v_1} = 0 \tag{B-21}$$

$$C_{31} = \frac{\partial \dot{x}}{\partial \phi} = -\ (v_c + \Delta v_1)\ \cos\ \phi \tag{B-22}$$

$$C_{32} = \frac{\partial \dot{x}}{\partial \Delta v_1} = -\sin\ \phi \tag{B-23}$$

$$C_{41} = \frac{\partial \dot{y}}{\partial \phi} = -(v_c + \Delta v_1)\ \sin\ \phi \tag{B-24}$$

$$C_{42} = \frac{\partial \dot{y}}{\partial \Delta v_1} = \ \cos\ \phi \tag{B-25}$$

Appendix C

Computer Program to Calculate Transfer

Trajectory to Wheeler's Orbit

This Appendix contains the computer routines used in this report. The computer language utilized was Fortran Extended Version IV, and all work was accomplished on the AFIT CDC 6600 computers. ODE was the integration package used in the program. For the integration steps, one time unit corresponds to 4.699939084 days.

```
      PROGRAM PROBA(INPUT,OUTPUT,TAPE6=OUTPUT,PLOT)
      EXTERNAL F
      DIMENSION X(20),WORK(520),IWORK(5),PHI(4,4),XIC(4),ID(17)
      DIMENSION XO(201),YO(201),WKAREA(14),E(2,1),C(4,2),D(2,4),B
(2,2),B
     CINV(2,2),DINPUT(2,1),TP(4,1)
      REAL MS
      COMMON THDOT,APDOT,G,SMU,MS,RS,RB,RSE
      DATA ID(1)/20HTRANSFER  TRAJECTORY/
      DATA ID(3)/20H     TO      ORBIT    /
      DATA ID(5)/20H   AROUND      L5     /
      DATA ID(7)/20H                      /
      DATA ID(9)/20H        X-AXIS        /
      DATA ID(11)/20H        Y-AXIS        /
      DATA ID(13)/50HMINIMUM DELTAV TRANSFER TO POINT 13(T=.81681
4089) /
      X(1)=-.5815938192003
      X(2)=.9928700556583
      X(3)=.2536637378670
      X(4)=.1850377439880
      TP(1,1)=X(1)$TP(2,1)=X(2)$TP(3,1)=X(3)$TP(4,1)=X(4)
      SMU=.0121396054
      TOL=.100E-12$REL=.100E-12
      WRITE (6,202) SMU,TOL,REL
202   FORMAT (/,1X,"MU= ",F12.10,10X,"TOL= ",E10.3,10X,"REL= ",E1
0.3,//)
      MS=328900.12
      WRITE (6,203) MS
203   FORMAT (1X,"MASS OF THE SUN = ",F9.2,/)
      DO 20 I=1,4
         XIC(I)=X(I)
20       CONTINUE
      NEQN=20$T=.81681408991
```

```
C      ◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆
C      ◆
C      THE INITIAL CONDITIONS FOR THE TARGET ORBIT AND NEW
C      ITERATED INITIAL CONDITIONS HAVE NOW BEEN STORED
C      ◆
C      ◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆
       THDOT=1.0
       APDOT=(29.5305882/365.25636556)$G=(APDOT+THDOT)◆◆2
       RS=((G)/((APDOT◆(1.+MS))◆◆2))◆◆(1./3.)$RB=(RS◆MS)
       RSE=RB+RS
       R1=((X(1)-SMU)◆◆2+X(2)◆2)◆◆.5
       R2=((1.-SMU+X(1))◆◆2+X(2)◆◆2)◆◆.5
       R3=((X(1)+RSE◆COS(T))◆◆2+(X(2)-RSE◆SIN(T))◆◆2)
      C◆◆(.5)
       WRITE (6,205) T,X(1),X(2),X(3),X(4)
205    FORMAT (35X,"T= ",F15.11,//,10X,"X= ",G20.13,20X,"Y= ",G20.
13,//,
      C10X,"XDOT= ",G20.13,17X,"YDOT= ",G20.13,/////)
       DO 21 J=5,20
          X(J)=0.0
21        CONTINUE
       DO 22 J=1,4
          X(5◆J)=1.0
22        CONTINUE
       DO 24 J=1,4
          DO 23 I=1,4
             PHI(J,I)=X(4◆J+I)
23           CONTINUE
24        CONTINUE
       IFLAG=1
       JCOUNT=0
       DO 26 K=101,201
          TOUT=T+.06283185307
          CALL ODE(F,NEON,X,T,TOUT,REL,REL,IFLAG,WORK,IWORK)
          JCOUNT=JCOUNT+1
```

```
          XD(K)=X(1)$YD(K)=X(2)
26      CONTINUE
        FEE=.7200775966$DELTAV=3.057381713
        RD=.0170110431$VC=8.2280688890
        WRITE(6,201) FEE,RD,DELTAV,VC
201     FORMAT(/,1X,"PHI= ",F12.10,10X,"RD= ",F12.10,/,1X,"DELTA V=
",F17.
       C10,10X,"ORBITAL VELOCITY = ",F12.10,//)
        TOL=.10E-1$REL=.10E-1
        WRITE (6,202) SMU,TOL,REL
204     FORMAT(1X,"THIS IS AN INTEGRATION ROUTINE THAT GENERATES A
       C",/,1X,"TRANSFER TRAJECTORY TO ORBIT AROUND L5",//)
        WRITE(6,301)TP(1),TP(2)
301     FORMAT(/,1X,"TARGET COORDINATES ARE ",/,10X,"X= ",G20.13,10
X,
       C"Y= ",G20.13,//)
        WRITE(6,302)TP(3,1),TP(4,1)
302     FORMAT(/,1X,"THE TARGET VELOCITIES ARE ",//,10X,"XDOT = ",G
20.13,
       C7X," YDOT = ",G20.13,//)
        TOF1=13000.0E-04$TOF2=21000.0E-04
        TOF=(TOF1+TOF2)/2
        TSTEP=TOF/100
        WRITE(6,303)TOF
303     FORMAT(1X,"TIME OF FLIGHT IS ",10X,G20.13,//)
        DO 41 J=5,20
           X(J)=0.0
41         CONTINUE
        DO 42 J=1,4
           X(5*J)=1.0
42         CONTINUE
C       ********************
C       *
C       THE INITIAL PHI MATRIX, PHI(TO,TO)=I
C       *
C       ********************
```

```
1070=        DO 44 J=1,4
1080=           DO 43 I=1,4
1090=              PHI(J,I)=X(4*J+I)
1100=43            CONTINUE
1110=44         CONTINUE
1120=        WRITE (6,208)
1130=208     FORMAT (/,1X,"*********************************************",/)
1140=40      X(1)=RO*COS(FEE)+SMU
1150=        X(2)=RO*SIN(FEE)
1160=        X(3)=-(VC+DELTAV)*SIN(FEE)
1170=        X(4)=(VC+DELTAV)*COS(FEE)
1180=        C(1,1)=-RO*SIN(FEE)
1190=        C(2,1)=RO*COS(FEE)
1200=        C(3,1)=-(VC+DELTAV)*COS(FEE)
1210=        C(4,1)=-(VC+DELTAV)*SIN(FEE)
1220=        C(1,2)=0.0
1230=        C(2,2)=0.0
1240=        C(3,2)=-SIN(FEE)
1250=        C(4,2)=COS(FEE)
1260=        DO 45 I=1,4
1270=           XIC(I)=X(I)
1280=45         CONTINUE
1290=        NEQN=20$T=.81681408991-TOF
1300=C       *********************
1310=C       *
1320=C       THE INITIAL CONDITIONS FOR THE TARGET ORBIT AND NEW
1330=C       ITERATED INITIAL CONDITIONS HAVE NOW BEEN STORED
1340=C       *
1350=C       *********************
1360=        THDOT=1.0
1370=        APDOT=(29.5305882/365.25636556)$G=(APDOT+THDOT)**2
1380=        RS=((G)/((APDOT*(1.+MS))**2))**(1./3.)$RB=(RS*MS)
1390=        RSE=RB+RS
1400=        R1=((X(1)-SMU)**2+X(2)**2)**.5
1410=        R2=((1.-SMU+X(1))**2+X(2)**2)**.5
1420=        R3=((X(1)+RSE*COS(T))**2+(X(2)-RSE*SIN(T))**2)
```

```
      C**(.5)
       IFLAG=1
       JCOUNT=0
       DO 46 K=1,100
          TOUT=T+TSTEP
          CALL ODE(F,NEQN,X,T,TOUT,REL,REL,IFLAG,WORK,IWORK)
          JCOUNT=JCOUNT+1
          XO(K)=X(1)$YO(K)=X(2)
46        CONTINUE
       DO 31 KK=1,4
          DO 30 JJ=1,4
          PHI(KK,JJ)=X(4*KK+JJ)
30           CONTINUE
31        CONTINUE
       DO 33 I=1,2
          DO 32 J=1,4
             D(I,J)=PHI(I,J)
32           CONTINUE
33        CONTINUE
       E(1)=X(1)-TP(1)
       E(2)=X(2)-TP(2)
       IF(ABS(E(1)).LT.TOL.AND.ABS(E(2)).LT.TOL) GO TO 27
       L=2$M=4$NM=2$J=1
       CALL MMULT(D,C,B,L,M,NM)
       IA=2$IDGT=2
       CALL LINV2F(B,NM,IA,BINV,IDGT,WKAREA,IER)
       MM=2$NN=1
       CALL MMULT(BINV,E,DINPUT,L,MM,NN)
       FEE=FEE-DINPUT(1)
       DELTAV=DELTAV-DINPUT(2)
314    FORMAT(/,1X,"FEE= ",G20.13,20X,"DELTAV= ",G20.13,/)
       GO TO 40
27     XD=TP(3,1)-X(3)$YD=TP(4,1)-X(4)
       DV=(XD**2+YD**2)**.5
       TDV=DV+DELTAV
       WRITE(6,214)
```

```
214   FORMAT(1X,"INITIAL LAUNCH CONDITIONS ARE",//)
      WRITE(6,314) FEE,DELTAV
      WRITE(6,315)
315   FORMAT(1X,"FINAL VELOCITIES ARE",//)
      WRITE(6,316)X(3),X(4)
316   FORMAT(10X,"XDOT = ",G20.13,15X,"YDOT = ",G20.13,//)
      WRITE(6,317)TDV
317   FORMAT(1X,"TOTAL DELTA V IS ",10X,G20.13,//)
      N=201$NO=1$NP=0$NS=1
      CALL HGRAPH(XO,YO,N,ID,NO,NP,NS)
      CALL PLOTE(MM)
      STOP "FINISHED"
      END
C
C
C     ****************************************************
C
C
      SUBROUTINE HGRAPH(X,Y,N,ID,NO,NP,NS)
      DIMENSION X(1),Y(1),ID(1) $ IF(NO.EQ.2) GO TO 30
      IF (NO.LT.0) GO TO 10
      CALL SCALE(X,7.,N,1)      $ CALL SCALE(Y,5.,N,1)
10    CALL PLOT(8.5,0.,-3) $   CALL PLOT(0.,11.,3)
      CALL PLOT(-1.35,1.35,3)
      CALL PLOT(-7.15,1.35,2)  $  CALL PLOT(-7.15,9.65,2)
      IF(ID(1).EQ.000) GO TO 25
      CALL PLOT(-7.05,9.55,3)    $ CALL PLOT(-7.05,7.55,2)
      DO 20 I=1,7,2
20    CALL SYMBOL(I*.1-6.9,7.85 ,.07,ID(I),90.,20)
      CALL PLOT(-7.05,7.55,3)    $ CALL PLOT(-6.05,7.55,2)
```

```
2090=       CALL PLOT(-6.05,9.55,2)     $ CALL PLOT(-7.05,9.55,2)
2100=       CALL PLOT(-7.15,9.65,3)
2110=25     CALL PLOT(-1.35,9.65,2)    $ CALL PLOT(-1.35,1.35,2)
2120=       CALL SYMBOL(-6.65,1.15,.1,ID(.3),0.,50)
2130=       CALL AXIS(-1.85,2.1,ID(9),-20,7.,90.,X(N+1),X(N+2))
2140=       CALL AXIS(-1.85,2.1,ID(11),20,5.,180.,Y(N+1),Y(N+2))
2150= 30    Y(N+2)=-Y(N+2)
2160=       X(N+1)=X(N+1)-2.1*X(N+2)  $ Y(N+1)=Y(N+1)+1.85*Y(N+2)
2170=       CALL LINE(Y,X,N,1,NP,NS)
2180=       X(N+1)=X(N+1)+2.1*X(N+2)  $ Y(N+1)=Y(N+1)-1.85*Y(N+2)
2190=       Y(N+2)=-Y(N+2)
2200=       RETURN                         $ END
2210=       SUBROUTINE VGRAPH(X,Y,N,ID,ND,NP,NS)
2220=       DIMENSION X(1),Y(1),ID(1)  $ IF(ND.EQ.2) GO TO 30
2230=       IF  (ND.LT.0) GO TO 10
2240=       CALL SCALE(X,4.9,N,1)       $ CALL SCALE(Y,7.0,N,1)
2250=10     CALL PLOT(8.5,0.,-3) $    CALL PLOT(0.,11.,3)
2260=       CALL PLOT(-1.35,1.35,3)
2270=       CALL PLOT(-7.15,1.35,2)   $  CALL PLOT(-7.15,9.65,2)
2280=       CALL PLOT(-1.35,9.65,2) $ IF(ID(1).EQ.000) GO TO 25
2290=       CALL PLOT(-1.45,9.55,3) $ CALL PLOT(-3.45,9.55,2)
2300=       DO 20 I= 1,7,2
2310=20     CALL SYMBOL(-3.15,9.4-I*.10,.07,ID(I),0.,20)
2320=       CALL PLOT(-3.45,9.55,3) $ CALL PLOT(-3.45,8.55,2)
2330=       CALL PLOT(-1.45,8.55,2) $ CALL PLOT(-1.45,9.55,2)
2340=       CALL PLOT(-1.35,9.65,3)
2350=25     CALL PLOT(-1.35,1.35,2)
2360=       CALL SYMBOL(-6.65,1.15,.1,ID(13),0.,50)
2370=       CALL AXIS(-6.4,1.85,ID(9),-20,4.9,0.,X(N+1),X(N+2))
2380=       CALL AXIS(-6.4,1.85,ID(11),20,7.0,90.,Y(N+1),Y(N+2))
2390= 30    X(N+1)=X(N+1)+6.4*X(N+2)   $ Y(N+1)=Y(N+1)-1.85*Y(N+2)
2400=       CALL LINE(X,Y,N,1,NP,NS)
2410=       X(N+1)=X(N+1)-6.4*X(N+2)   $ Y(N+1)=Y(N+1)+1.85*Y(N+2)
2420=       RETURN      $ END
2430=C
```

```
2440=C
2450=        SUBROUTINE F(T,X,XP)
2460=        DIMENSION X(20),XP(20),A(4,4),PHI(4,4),PHIDOT(4,4)
2470=        REAL MS
2480=        COMMON THDOT,APDOT,G,SMU,MS,RS,RB,RSE
2490=        R1=((X(1)-SMU)**2+X(2)**2)**.5
2500=        R2=((1.-SMU+X(1))**2+X(2)**2)**.5
2510=        R3=((X(1)+RSE*COS(T))**2+(X(2)-RSE*SIN(T))**2)**(.5)
2520=C       ********************************************
2530=C       *
2540=C       THE A MATRIX EQUATIONS ARE BELOW
2550=C       *
2560=C       ********************************************
2570=        A(1,1)=0
2580=        A(1,2)=0
2590=        A(2,1)=0
2600=        A(2,2)=0
2610=        A(1,4)=0
2620=        A(2,3)=0
2630=        A(3,3)=0
2640=        A(4,4)=0
2650=        A(1,3)=1.0
2660=        A(2,4)=1.0
2670=        A(3,4)=2.*(THDOT+APDOT)
2680=        A(4,3)=-2.*(THDOT+APDOT)
2690=        A(3,1)=(THDOT+APDOT)**2-((1.-SMU)*G)/(R1**3)-(G*SMU)/(R2**3
)+
  2700=     ,((3.*(1.-SMU)*G*((X(1)-SMU)**2))/(R1**5))+(3.*SMU*G*((1.-SM
U+X(1))
  2710=     ,**2))/(R2**5)-(G*MS)/(R3**3)+(3.*((X(1)+RSE*COS(T))**2)*MS*
G)/
  2720=     ,(R3**5)
  2730=      A(3,2)=(3.*X(2)*SMU*G*(1.-SMU+X(1)))/(R2**5)+(3.*X(2)*(X(1)
-SMU)
  2740=     +*(1.-SMU)*G)/(R1**5)+(3.*(X(2)-RSE*SIN(T))*(X(1)+RSE*COS(T)
```

```
)
      ,*MS*G)/(R3**5)
      A(4,1)=A(3,2)
      A(4,2)=(THDOT+APDOT)**2-((1.-SMU)*G)/(R1**3)-(SMU*G)/(R2**3
)+
      ,(3.*(1.-SMU)*(X(2)**2)*G)/(R1**5)+(3.*SMU*(X(2)**2)*G)/(R2*
*5)-
      ,(G*MS)/(R3**3)+(3.*((X(2)-RSE*SIN(T))**2)*MS*G)/(R3**5)
      DO 30 J=1,4
      DO 40 I=1,4
      PHI(J,I)=X(4*J+I)
   40 CONTINUE
   30 CONTINUE
      CALL MMULT(A,PHI,PHIDOT,4,4,4)
C     ********************************************
C     *
C     THE FOLLOWING EQUATIONS ARE THE 16 PHIDOT EQUATIONS
C     TO BE INTEGRATED
C     *
C     ********************************************
      DO 80 JJ=1,4
      DO 90 II=1,4
      XP(4*JJ+II)=PHIDOT(JJ,II)
   90 CONTINUE
   80 CONTINUE
C     ********************************************
C     *
C     THE FOLLOWING EQUATIONS ARE THE 4 STATE EQUATIONS TO BE INT
EGRATED
C     *
C     ********************************************
      XP(1)=X(3)
      XP(2)=X(4)
      XP(3)=2.*X(4)*(THDOT+APDOT)+X(1)*((THDOT+APDOT)**2)+RB*(APD
DT**2)
```

```
      ,◆COS(T)-(((X(1)-SMU)◆(1.-SMU)◆G)/(R1◆◆3))-(((1.-SMU+X(1))◆S
MU◆G)/
      ,(R2◆◆3))-(((X(1)+RSE◆COS(T))◆MS◆G)/(R3◆◆3))
       XP(4)=-2.◆X(3)◆(THDOT+APDOT)+X(2)◆((THDOT+APDOT)◆◆2)-RB◆(AP
DOT◆◆2)
      ,◆SIN(T)-((X(2)◆(1.-SMU)◆G)/(R1◆◆3))-((X(2)◆SMU◆G)/(R2◆◆3))-
      ,(((X(2)-RSE◆SIN(T))◆MS◆G)/(R3◆◆3))
       RETURN

       END
       SUBROUTINE MMULT(A,B,C,L,M,N)
       DIMENSION A(L,M),B(M,N),C(L,N)
C      ◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆
C      ◆
C      THIS SUBROUTINE MULTIPLIES A LXM MATRIX BY A MXN MATRIX
C      ◆
C      ◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆◆
       DO 50 K=1,L
       DO 60 J=1,N
       C(K,J)=0.0
       DO 70 KK=1,M
       C(K,J)=A(K,KK)◆B(KK,J)+C(K,J)
   70  CONTINUE
   60  CONTINUE
   50  CONTINUE
       RETURN
       END
◆EOR
◆EOF
```

Vita

Theron E. Weimer was born 8 August 1948 in Cedar Rapids, Iowa. He graduated from Thomas Jefferson High School, Cedar Rapids, Iowa, June 1966. The next fall he entered the University of Iowa, Iowa City, Iowa. After one year of college he received an appointment to the United States Air Force Academy and entered June 1967. He graduated from the USAFA June 1971 with a Bachelor of Science degree in Aeronautical Engineering and a regular commission as a 2nd Lieutenant in the United States Air Force. His initial assignment was to Undergraduate Pilot Training (UPT) at Williams Air Force Base, Arizona. Upon graduation from UPT in June 1972 as an Air Force pilot, he was assigned to the 316th Tactical Airlift Wing, Langley Air Force Base, Virginia as a C-130E co-pilot. While there he upgraded to the position of Aircraft Commander and also performed as an airlift mission scheduler.

In July 1975 he was reassigned to the 35th Training Squadron, Reese Air Force Base, Texas, as a T-37B instructor pilot. During the assignment at Reese he was an assistant flight commander and a Standardization/Evaluation Flight Examiner. He also earned a Master of Science degree in Business Management from the University of Northern Colorado while at Reese Air Force Base.

In June 1978 he was assigned to the Air Force Institute of Technology's resident School of Engineering at Wright-Patterson Air Force Base, Ohio and began his studies toward a Master of Science degree in Astronautical Engineering. Upon graduation, he will be assigned to the Combat Operations Center of the Cheyenne Mountain Complex, Colorado Springs, Colorado.

Permanent Address: 1615 13th Street N.W.
Cedar Rapids, Iowa 52405

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: AFIT/GA/AA/79D-11	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): DETERMINATION OF THE MINIMUM DELTA V TRANSFER TRAJECTORY FROM A LOW EARTH ORBIT TO A STABLE ORBIT AROUND THE LAGRANGIAN POINT L4 IN A RESTRICTED FOUR-BODY SYSTEM	5. TYPE OF REPORT & PERIOD COVERED: MS Thesis
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Theron E. Weimer, Captain	8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Air Force Institute of Technology (AFIT-EN) Wright-Patterson AFB, Ohio 45433	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE: December, 1979
	13. NUMBER OF PAGES: 110
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Approved for public release; IAW AFR 190-17

Joseph P. Hipps, Major, USAF
Director of Public Affairs, A.F.I.T.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Transfer Trajectory
L4

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this report, the equations of motion for a satellite in restricted four-body motion are derived and examined. An algorithm is developed that produces a minimum delta v transfer trajectory from a low Earth orbit to establishment in Wheeler's stable periodic orbit around L4. Both a Hohmann transfer and an infinite velocity, straight-line transfer are examined as initial conditions for integration of the equations of motion. Transfer trajectories are determined to ten points around Wheeler's orbit so that a

DD FORM 1473, 1 JAN 73 — EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

minimum delta v trajectory can be determined. Two types of minimum delta v values were found. One type is a true minimum value, the other type is a limiting value beyond which the target orbit can not be reached. Results are presented on a transfer trajectory that gives a minimum delta v from the points examined. From these results a more accurate transfer location is estimated.